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The $1/\mathrm{N}$ expansion method in quantum field theory

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Introduction and motivations

Methods of resolution of quantum field theory equations are very rare. Usually one uses perturbation theory with respect to the coupling constant g starting from free field theory. The size of (dimensionless) g gives an estimate of the strength of the interaction.

The presence in the theory of other parameters than the coupling constant may allow use of perturbation theory with respect to those parameters. This may enlarge the possibilities of approximate resolutions of the theory.

A new parameter may emerge if the system under consideration satisfies symmetry properties with respect to a group of internal transformations.

For example, constituents of the system (particles, nucleons, nuclei, energy excitations, etc.) belonging to different species, may have the same masses and possess the same dynamical properties with respect to the interaction. In such a case interchanges between these constituents would not modify the physical properties of the system. The latter interchanges might also be considered in infinitesimal or continuous forms like for the rotations in ordinary space. The system then satisfies an invariance property under continuous symmetry transformations.

In many cases, the invariance might also be only approximate.

Among the continuous symmetry groups, two play an important role in physical problems.

The first is $O(N)$, the orthogonal group, generated by the $N \times N$ orthogonal matrices. It has $N(N-1)/2$ parameters and generators.

The second is $SU(N)$, the unitary group, generated by the $N \times N$ unitary complex matrices, with determinant $= 1$. It has $(N^2 - 1)$ parameters and generators.

In particle and nuclear physics, one has the approximate isospin symmetry group $SU(2)$ and the approximate flavor symmetry group $SU(3)$. Using these approximations, one can establish relations between masses and physical parameters of various particles, prior to solving the dynamics of the system under consideration.

In the cases of the groups $O(N)$ and $SU(N)$ mentioned above, N has a well defined fixed value in each physical problem, $N = 2, 3, \ldots$, etc.

It is however tempting to consider the case where N is a free parameter which can be varied at will. In particular, large values of N, with the limit $N \to \infty$, seem to be of interest.

At first sight, it might seem that taking large values of N would lead to more complicated situations, since the number of parameters increases and the group representations become intricate. However, it was noticed that when the limit is taken in an appropriate way, in conjunction with the coupling constant of the theory, it may lead to simpler results than the cases of finite N.

If this happens, then an interesting perspective of resolution opens.

One may solve the problem in the simplified situation of the limit $N \to \infty$ and then, to improve the predictions, consider the contributions of the terms of order $O(1/N)$ as a perturbation.

If the true N is sufficiently large, then the zeroth-order calculation done in the limit $N \to \infty$ would already provide the main dominant aspects of the solution of the physical problem under consideration. This is the spirit of the $O(1/N)$ expansion method in quantum field theory.

We will illustrate the method by two explicit examples:
1) The $O(N)$ model of scalar fields.
2) The two-dimensional model of Gross–Neveu with global $U(N)$ symmetry of fermion fields and discrete chiral symmetry.

Historical background

The first who noticed the interest of the large-N limit was Stanley in statistical physics (1968), who used it in the framework of the Heisenberg model (spin-spin interactions). He showed that in that limit the model reduces to the spherical approximation of the Ising model, which was soluble.

H. E. Stanley, Phys. Rev. 176 (1968) 718.

The first who introduced the method in quantum field theory is Wilson (1973), who was aware of the work of Stanley in statistical physics. He applied it to the $O(N)$ model of scalar fields and to the $U(N)$ model of fermion fields.

K. Wilson, Phys. Rev. D 7 (1973) 2911.

In 1974, 't Hooft applied it to Quantum Chromodynamics (QCD), the newly born theory of the strong interaction, which is a gauge theory with the non-Abelian gauge group $SU(3)_c$ in the internal space of color quantum numbers. (Not to confound with the global flavor symmetry group $SU(3)$ met previously.)

This theory cannot be solved with the only large-N limit, but many simplifications occur. In two space-time dimensions, it is almost soluble.

G. 't Hooft, Nucl. Phys. B72 (1974) 461.

G. 't Hooft, Nucl. Phys. B75 (1974) 461.

Reviews

E. Witten: Baryons in the $1/N$ expansion,

Nucl. Phys. B160 (1979) 57.

E. Witten: The $1/N$ expansion in atomic and particle physics,

Lectures at the Cargese Summer School, *"Recent Developments in Gauge Theories"*, 1979, edited by G. 't Hooft *et al.* (Plenum, N.Y., 1980).

S. Coleman: $1/N$,

Lectures at the Erice International School of Subnuclear Physics, 1979 (Plenum, N.Y., 1980), p. 805. Also in *"Aspects of Symmetry"* (Cambridge University Press, 1985), p. 351.

Yu. Makeenko: Large-N gauge theories,

Lectures at the 1999 NATO-ASI on *"Quantum Geometry"* in Akureyri, Iceland; arXiv: hep-th/0001047.

The scalar $O(N)$ model

The $O(N)$ model is a theory of N real scalar fields ϕ^a ($a = 1, 2, \ldots, N$), with a quartic interaction, invariant under the $O(N)$ group of transformations. This group is similar in structure to the rotation group, but acts in the internal space of N species of the fields.

$$\mathcal{L} = \frac{1}{2}\, \partial_\mu \phi^a \partial^\mu \phi^a - \frac{1}{2}\, \mu_0^2 \phi^a \phi^a - \frac{\lambda_0}{8N}(\phi^a \phi^a)^2.$$

(Summation on repeated indices is understood. $\partial_\mu = \frac{\partial}{\partial x^\mu}$, etc.) μ_0^2 and λ_0 are real parameters, representing the bare mass squared and the bare coupling constant. In four space-time dimensions the coupling constant is dimensionless.

S. Coleman, R. Jackiw and H. D. Politzer, Phys. Rev. D 10 (1974) 2491.

Search for the ground state of the theory. The lagrangian density is composed of the kinetic energy density minus the potential energy density. For constant fields, the kinetic energy vanishes. The potential energy density is:

$$U = \frac{1}{2}\,\mu_0^2 \phi^a \phi^a + \frac{\lambda_0}{8N}(\phi^a \phi^a)^2.$$

Quartic function of the ϕs. It is evident that the energy is not bounded from below if $\lambda_0 < 0$. In this case no stable ground state.
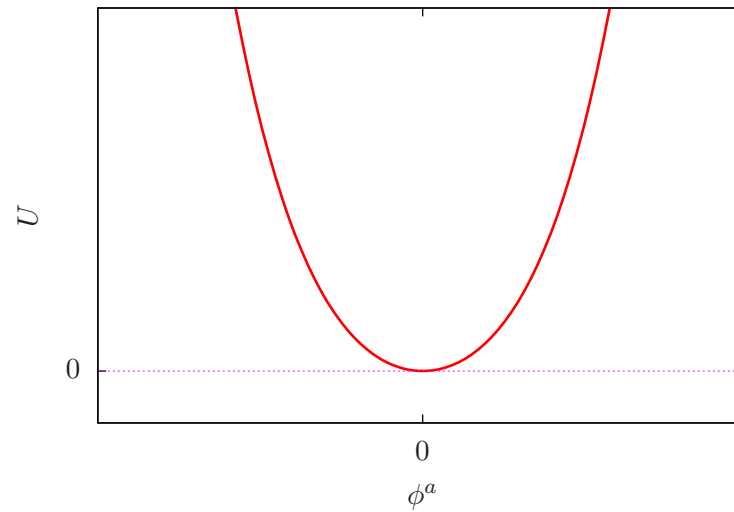
We assume henceforth that $\lambda_0 > 0$.

$\lambda_0 > 0$.

Two cases have to be distinguished: 1) $\mu_0^2 > 0$; 2) $\mu_0^2 < 0$.

1) $\mu_0^2 > 0$.

Shape of U.

The minimum of the energy corresponds to the values $\phi^a = 0$, $(a = 1, \ldots, N)$. The ϕs in this case can be considered as excitations from the ground state.

Considering the lagrangian, we can interpret it as describing the dynamics of N particles with degenerate masses, equal to μ_0, interacting by means of the quartic interaction. We have complete $O(N)$ symmetry between the particles.

This mode of symmetry realization is called the Wigner mode or the normal mode.

2) $\mu_0^2 < 0$.

Shape of U.
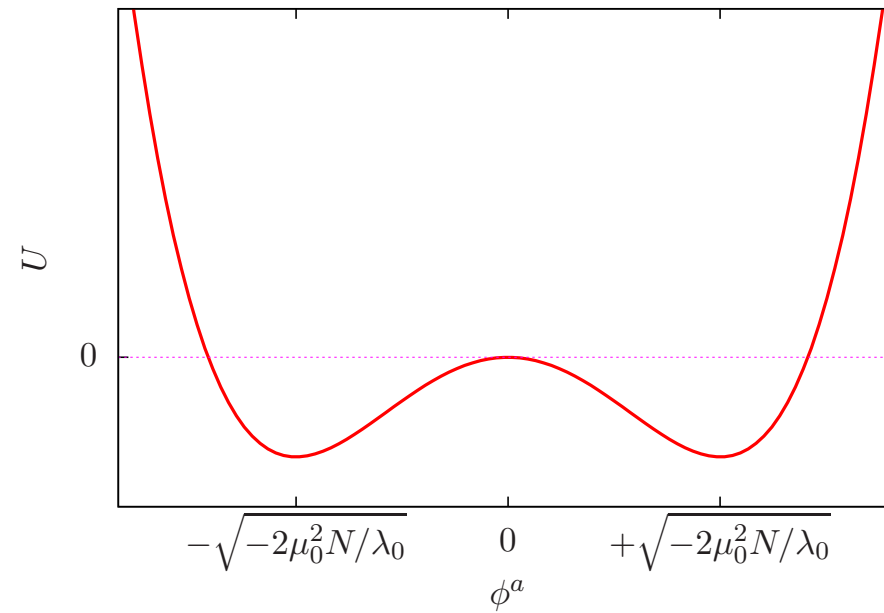
The minimum of U is no longer $\phi^a = 0$, but a shifted value at $\phi^a = \pm\sqrt{-\frac{2\mu_0^2 N}{\lambda_0}} \equiv\; <\phi>$ for one of the as. There is a degeneracy between the different ϕ^as.

Redefine the fields around the new ground state.

$$\chi = \phi^N - <\phi>, \qquad \pi^a = \phi^a, \quad a = 1,\ldots,N-1.$$

the potential energy becomes

$$U = -\mu_0^2\chi^2 + \frac{\lambda_0}{2N}<\phi>\chi^3 + \frac{\lambda_0}{8N}(\pi^a\pi^a + \chi^2)^2.$$

The fields π^a have no longer mass terms, while the field χ has a mass term. We have now the masses:

$$m_{\pi^a}^2 = 0, \quad a = 1,\ldots,N-1, \quad m_\chi^2 = -2\mu_0^2.$$

The $O(N)$ symmetry that we had initially has partially disappeared. There is now $O(N-1)$ symmetry in the space of the fields π^a.

It is said that the $O(N)$ symmetry has been spontaneously broken. This happened because the ground state of the theory is not symmetric, while the lagrangian is.

This is accompanied with the appearance of $(N-1)$ massless fields. These are called Goldstone bosons.

This way of realization of the symmetry is called the Goldstone mode.

In particle and nuclear physics, the isospin $SU(2)$ symmetry and the quark flavor $SU(3)$ symmetry are realized with the Wigner mode.

The chiral $SU(3)_R \times SU(3)_L$ symmetry is realized with the Goldstone mode.

Quantum effects

We want now to take into account the quantum effects of the model that we are considering. For this, it is necessary to compute the quantum corrections that contribute to the potential energy. In quantum field theory, these are represented by the radiative corrections.

The definition of U is enlarged. The new potential energy is called the effective potential.

$$U_{eff} = U_{class} + U_{rad} \,.$$

U_{rad} = sum of all connected loop diagrams with external constant fields.

The calculation of U_{rad} is almost impossible in exact form in the general case: there are infinities of complicated diagrams to calculate.

However, the use of the $1/N$ expansion method simplifies considerably the situation. The leading terms of this expansion are calculable.

Propagators, vertices, loops

The radiative corrections can be calculated starting from the situation where $\mu_0^2 > 0$. Radiative corrections will modify the value of μ_0^2 into a new value μ^2. It is then sufficient to study at the end the analytic continuation of μ^2 into negative values.

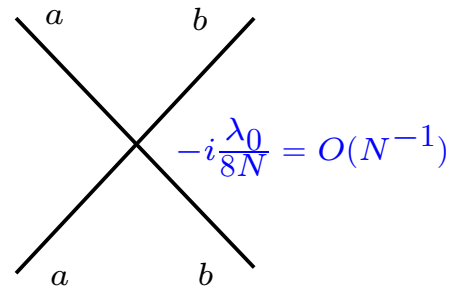We therefore consider the initial lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi^a\partial^\mu\phi^a - \frac{1}{2}\,\mu_0^2\phi^a\phi^a - \frac{\lambda_0}{8N}(\phi^a\phi^a)^2.$$

The free propagator of the field ϕ^a is obtained from the quadratic parts of $\mathcal{L}$ (not containing the coupling constant λ_0)
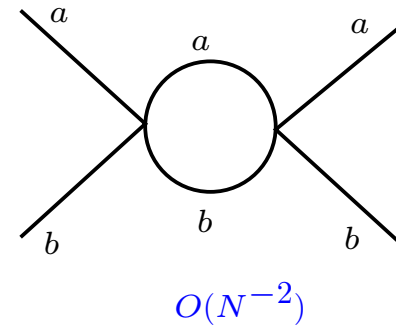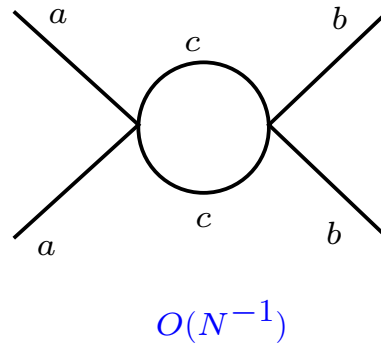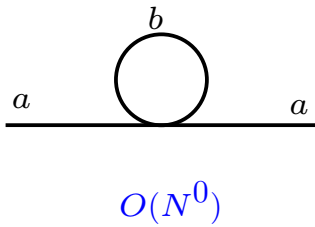
$$D_0^{ab}(p) = \delta_{ab}D_0(p) = \int d^4x e^{ip.x} < 0|T(\phi^a(x)\phi^b(0))|0 >,$$

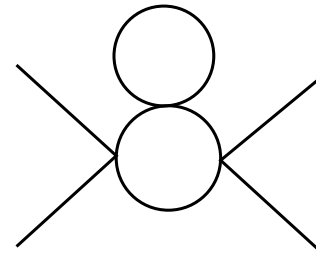$$D_0(p) = \frac{i}{p^2 - \mu_0^2 + i\varepsilon} \qquad \Longleftrightarrow \qquad \underline{\hspace{2cm}}$$

The vertex is represented by the four-field interaction term (contact interaction).

$$-i\frac{\lambda_0}{8N} = O(N^{-1})$$

Radiative corrections are represented by loops.

$O(N^0)$

$O(N^{-1})$

$O(N^{-2})$

Examples of two-loop diagrams.
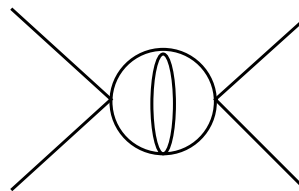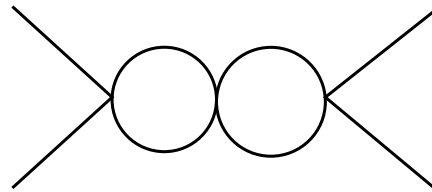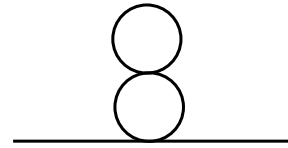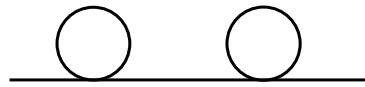
The auxiliary field method

Simplify the analysis of the $1/N$ expansion by introducing an auxiliary field describing collectively the composite field $\phi^a \phi^a$.
Add to the lagrangian a new term:

$$\mathcal{L}' = \mathcal{L} + \frac{N}{2\lambda_0} \left(\sigma - \frac{\lambda_0}{2N} \, \phi^a \phi^a - \mu_0^2 \right)^2.$$

The σ field does not have kinetic energy. Its equation of motion yields its definition:

$$\sigma = \frac{\lambda_0}{2N} \, \phi^a \phi^a + \mu_0^2,$$

which shows that the added term in the lagrangian is zero. Thus the new lagrangian is equivalent to the former one:

$$\mathcal{L}' \approx \mathcal{L} \, .$$

$$\mathcal{L}' = \frac{1}{2}\,\partial_\mu \phi^a \partial^\mu \phi^a + \frac{N}{2\lambda_0}\sigma^2 - \frac{1}{2}\sigma\phi^a\phi^a - \frac{N\mu_0^2}{\lambda_0}\sigma.$$

Propagators:

$$D_{0\phi}(p) = \frac{i}{p^2 + i\varepsilon} \iff \underline{\hspace{3cm}} = O(N^0),$$

$$D_{0\sigma}(p) = \frac{i\lambda_0}{N} \iff \text{----------} = O(N^{-1}).$$

Vertex $\sigma\phi\phi$:

$$= \; O(N^0).$$

Higher-order diagrams.

$O(N^0)$

$O(N^0)$

$O(N^{-1})$

$O(N^{-1})$

$O(N^{-2})$
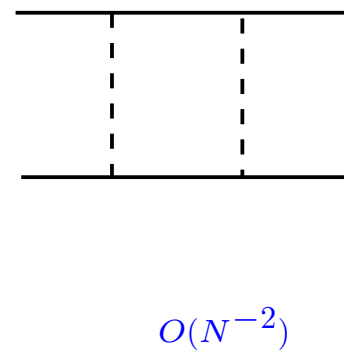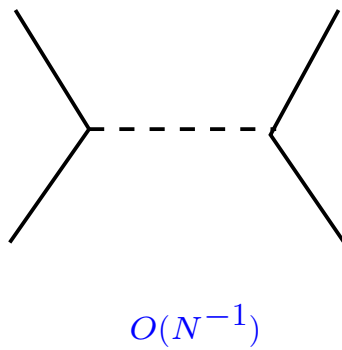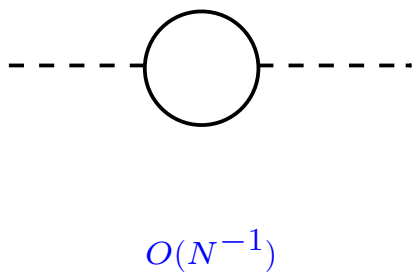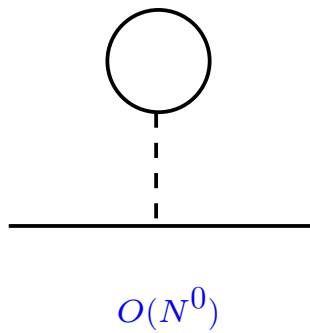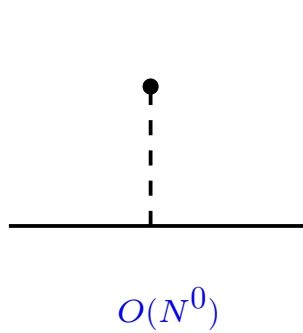
Important property: Internal $\boldsymbol{\sigma}$ propagators give each $O(N^{-1})$ contributions. $\Longrightarrow$ Their contributions are negiligible. One can ignore all radiative corrections (loops) containing such propagators.

The only contributions that survive in radiative corrections are one-loop diagrams of ϕ fields.

At leading-order in the $1/N$ expansion the theory is explicitly calculable.

Effective potential

$$U_{eff} = U_{class} + U_{rad} \, .$$

$$U_{class} = -\frac{N}{2\lambda_0}\sigma^2 + \frac{1}{2}\sigma\phi^a\phi^a + \frac{N\mu_0^2}{\lambda_0}\sigma.$$

$$U_{rad} =$$

$+$ $+$ $+ \dots$
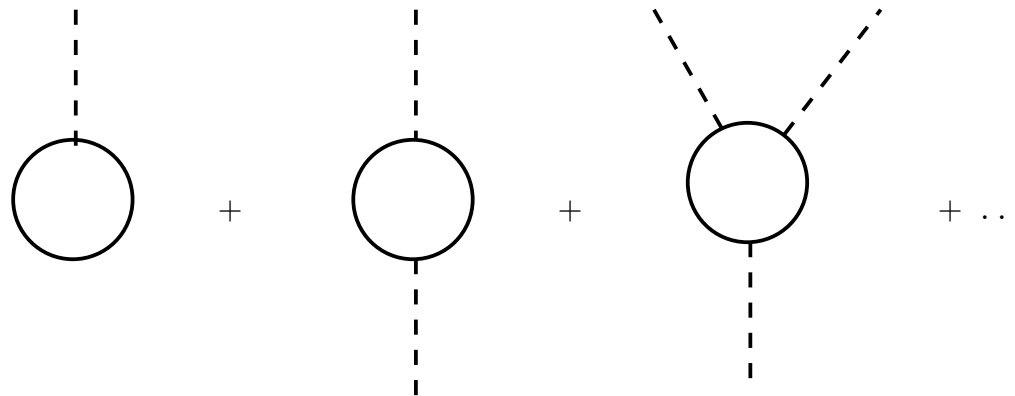
S. Coleman and E. Weinberg, Phys. Rev. D 7 (1973) 1888.

R. Jackiw, Phys. Rev. D 9 (1974) 1686.

$$U_{eff} = -\frac{N}{2\lambda_0}\sigma^2 + \frac{1}{2}\sigma\phi^a\phi^a + \frac{N\mu_0^2}{\lambda_0}\sigma + \frac{N}{2}\int\frac{d^4k}{(2\pi)^4}\ln(k^2+\sigma).$$

(Integral over euclidean momenta.)

We are interested by the stationary point of U_{eff} (ground state).

$$\frac{\partial U_{eff}}{\partial\sigma} = -\frac{N}{\lambda_0}\sigma + \frac{1}{2}\phi^a\phi^a + \frac{N\mu_0^2}{\lambda_0} + \frac{N}{2}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+\sigma)},$$

$$\frac{\partial U_{eff}}{\partial\phi^a} = \phi^a\sigma, \qquad a = 1,\ldots,N.$$

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The integral $\int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+\sigma)}$ is divergent for values of $k \to \infty$.

To cure this problem, one absorbs the divergent parts into the bare coupling constant and the mass, defining finite quantities.

Asymptotic behavior of the integrand:

$$\frac{1}{k^2 + \sigma} \underset{k^2 \to \infty}{\longrightarrow} \frac{1}{k^2} - \frac{\sigma}{(k^2)^2} + O(1/(k^2)^3).$$

The first two terms lead to ultraviolet divergences by integration. We cannot, however, subtract them as they stand, since the second term would lead to a new artificial infrared divergence.

We have to incorporate in the second term a mass factor in the denominator to render it softer in the infrared region.

We add and subtract in $\frac{\partial U_{eff}}{\partial \sigma}$ the following quantities:

$$\frac{N}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} - \frac{N}{2}\sigma \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k^2 + M^2)},$$

where M is an arbitrary mass term.

The result is

$$\frac{\partial U_{eff}}{\partial \sigma} = -N\sigma\left(\frac{1}{\lambda_0} + \frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2(k^2 + M^2)}\right) + \frac{1}{2}\phi^a\phi^a$$

$$+N\left(\frac{\mu_0^2}{\lambda_0} + \int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2}\right) + \frac{N}{32\pi^2}\sigma \ln(\frac{\sigma}{M^2}).$$

The infinite integrals, together with the bare coupling constant and the bare mass term, may define finite renormalized quantities.

Finite coupling constant and mass term:

$$\frac{1}{\lambda(M)} = \frac{1}{\lambda_0} + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2(k^2 + M^2)},$$

$$\frac{\mu^2(M)}{\lambda(M)} = \frac{\mu_0^2}{\lambda_0} + \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2}.$$

The finite quantities depend on the arbitrary mass parameter M. For each fixed value of M, they take a corresponding value.

One can better study this dependence by comparing for instance the value of the coupling constant $\lambda(M)$ to a reference value $\lambda_1 \equiv \lambda(M_1)$ corresponding to a reference mass parameter M_1.

Renormalization group equation (RGE)

One obtains from the comparison of the corresponding two equations:

$$\frac{1}{\lambda(M)} = \frac{1}{\lambda_1} - \frac{1}{32\pi^2} \ln(\frac{M^2}{M_1^2}).$$

This equation is called the renormalization group equation and plays an important role for the analysis and understanding of the properties of the theory.

$$\lambda(M) = \frac{\lambda_1}{1 - \frac{\lambda_1}{32\pi^2} \ln(\frac{M^2}{M_1^2})}.$$

When M increases starting from M_1, $\lambda(M)$ increases. When M reaches the value $M_{cr} = M_1 \exp(16\pi^2/\lambda_1)$, $\lambda(M)$ diverges. This is a sign of the instability of the theory for large values of λ.

The RGE can also be formulated in the form of a differential equation. Defining
$$M\frac{\partial \lambda(M)}{\partial M} = \beta(\lambda(M)),$$
one finds from the finite form of $\lambda(M)$

$$\beta(\lambda) = \frac{\lambda^2}{16\pi^2} > 0,$$

which shows that $\lambda(M)$ is an increasing function of M.

Physical interest of the RGE

In perturbation theory, when calculating radiative corrections, one usually finds powers of the quantity $\lambda^2 \ln(p^2/M^2)$, where p is a representative of the momenta of the external particles. Even if λ is chosen small, at high energies, i.e., at large values of p, the logarithm may become large enough to invalidate perturbative calculations.

However, since the logarithm depends on M for dimensional reasons, one can choose the latter of the order of p to maintain the logarithm small. This could be useful, if on the other hand λ remains bounded or small at these values of M. The RGE precisely gives us the answer to that question, indicating the way λ behaves at large values of M.

Coming back to our theory, we see from our previous results that this is not the case here: λ, on the contrary, increases with M and diverges at M_{cr}. The theory is ill-defined at high energies. In x-space, high energies correspond to short distances. The interaction becomes stronger when the distance between two sources or two particles decreases. A similar behavior is also found in Quantum Electrodynamics (QED). These theories are better defined at low energies.

Theories for which the coupling constant decreases at high energies or at short distances are called asymptotically free.

The ground state

The effective potential.

$$U_{eff} = \frac{1}{2}\phi^a\phi^a\sigma - \frac{N\sigma^2}{2\lambda}(1 + \frac{\lambda}{64\pi^2}) + \frac{N\mu^2\sigma}{\lambda} + \frac{N\sigma^2}{64\pi^2}\ln(\frac{\sigma}{M^2}).$$

Minimum of the effective potential.

$$\frac{\partial U_{eff}}{\partial\sigma} = -\frac{N\sigma_0}{\lambda} + \frac{1}{2}\phi_0^a\phi_0^a + \frac{N\mu^2}{\lambda} + \frac{N}{32\pi^2}\sigma_0\ln(\frac{\sigma_0}{M^2}) = 0,$$

$$\frac{\partial U_{eff}}{\partial\phi^a} = \phi_0^a\sigma_0 = 0, \qquad a = 1, \ldots, N.$$

Two possibilities.

1) $\phi_0^a = 0$ $(a = 1, \ldots, N)$, $\sigma_0 \neq 0$.

From the first equation one obtains

$$\sigma_0 \left(1 - \frac{\lambda}{32\pi^2} \ln\left(\frac{\sigma_0}{M^2}\right) \right) = \mu^2.$$

The presence of the logarithm imposes $\sigma_0 > 0$. For small values of λ, one has $\mu^2 > 0$. By continuity to larger values, one should search for positive solutions of σ_0. Let σ_0 be the solution of the equation for a certain domain of λ and μ^2. One should redefine σ from the value of σ_0:

$$\sigma' = \sigma - \sigma_0.$$

The corresponding shift of σ in the lagrangian gives back a common mass to the fields ϕ^a. We are in a situation where the $O(N)$ symmetry is realized in the Wigner mode.

2) $\sigma_0 = 0$.

From the first equation:

$$\frac{1}{2}\phi_0^a \phi_0^a + \frac{N\mu^2}{\lambda} = 0. \quad \Longrightarrow \quad \mu^2 < 0.$$

$$\phi_0^a \phi_0^a = -\frac{2N\mu^2}{\lambda}.$$

There is a degeneracy of solutions for the ϕ_0^as. One can choose for example:

$$\phi_0^a = 0, \quad a = 1, \ldots, N-1, \quad \phi_0^N = \sqrt{-\frac{2N\mu^2}{\lambda}} \equiv <\phi>.$$

One then develops ϕ^N around $<\phi>$:

$$\chi = \phi^N - <\phi>.$$

The symmetry is realized in the Goldstone mode with the presence of $(N-1)$ massless fields.

Summary

The radiative corrections introduce limitations into the validity domain of the model.

The coupling constant increases at high energies and diverges at some critical mass scale.

The ground state equations also introduce new constraints on the parameters of the model.

The two modes of realization of the symmetry, found at the classical level, remain valid within the restricted domain of the parameters.

These results are obtained at the leading-order of the $1/N$ expansion.

The Gross-Neveu model

This is a two-dimensional model of N fermions, interacting by means of a four-fermion interaction, satisfying global $U(N)$ symmetry. The system satisfies also a discrete chiral symmetry, which prevents the fermions from having a mass.

$$\mathcal{L} = \overline{\psi}^a i \gamma^\mu \partial_\mu \psi^a + \frac{g_0}{2N} (\overline{\psi}^a \psi^a)^2, \qquad g_0 > 0.$$

In two space-time dimensions, the Dirac fields have two components and the Dirac matrices γ reduce to the Pauli 2×2 matrices σ:

$$\gamma^0 = \sigma_z, \quad \gamma^1 = i\sigma_y, \quad \gamma_5 = \gamma^0 \gamma^1 = \sigma_x.$$

Discrete chiral transformations:

$$\psi^a \to \gamma_5 \psi^a, \qquad \overline{\psi}^a = \psi^{\dagger a} \gamma^0 \to -\overline{\psi}^a \gamma_5.$$

A mass term is not invariant under these transformations:

$$m\overline{\psi}^a \psi^a \to -m\overline{\psi}^a \psi^a.$$

Hence, the fermions should be massless.

The fermion field has mass dimension $1/2 \implies$ the coupling constant g_0 is dimensionless.

The auxiliary field

Introduction of the auxiliary field σ by adding to the lagrangian a defining term:

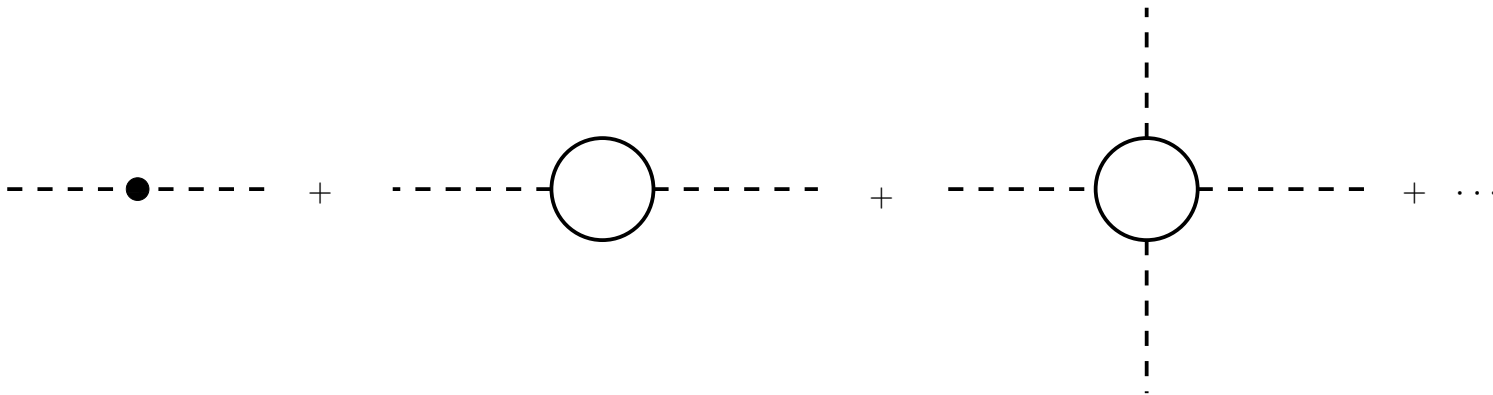$$\mathcal{L}' = \mathcal{L} - \frac{N}{2g_0}\left(\sigma + \frac{g_0}{N}\,\overline{\psi}^a\psi^a\right)^2.$$

$$\mathcal{L}' = \overline{\psi}^a i\gamma^\mu\partial_\mu\psi^a - \frac{N}{2g_0}\sigma^2 - \sigma\,\overline{\psi}^a\psi^a.$$

$$U_{eff} = U_{class} + U_{rad} \; .$$

Calculations are done with a constant value of the σ field, which corresponds to the vacuum expectation value of $\sigma(x)$, $< 0|\sigma(x)|0 >$, which is translation invariant and hence independent of x. The fermion fields have zero vacuum expectation values, hence do not contribute to the external fields of U_{eff}.

$$U_{eff} =$$

$$U_{eff} = N\left(\frac{\sigma^2}{2g_0} - \int \frac{d^2k}{(2\pi)^2} \ln(1 + \frac{\sigma^2}{k^2})\right),$$

$$\frac{\partial U_{eff}}{\partial \sigma} = 2\sigma N\left(\frac{1}{2g_0} - \int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \sigma^2}\right).$$

(The integration momenta are euclidean.) As for the scalar case, we have to isolate the divergent part of the integral. Again we must introduce a mass parameter in the subtracted term to avoid infrared divergence.

$$\frac{\partial U_{eff}}{\partial \sigma} = \sigma N\left[\frac{1}{g_0} - 2\int \frac{d^2k}{(2\pi)^2} \frac{1}{k^2 + \mu^2}\right] + \frac{\sigma N}{2\pi} \ln(\frac{\sigma^2}{\mu^2}).$$

We deduce the renormalization of the coupling constant into a finite value:

$$\frac{1}{g(\mu)} = \frac{1}{g_0} - 2 \int \frac{d^2 k}{(2\pi)^2} \frac{1}{k^2 + \mu^2}, \quad g > 0,$$

$$\frac{\partial U_{eff}}{\partial \sigma} = \sigma N \left[\frac{1}{g(\mu)} + \frac{1}{2\pi} \ln(\frac{\sigma^2}{\mu^2}) \right].$$

g dependes on the mass parameter μ. Its relation to another choice, μ_1, with $g(\mu_1) \equiv g_1$, is:

$$\frac{1}{g(\mu)} = \frac{1}{g_1} + \frac{1}{2\pi} \ln(\frac{\mu^2}{\mu_1^2}),$$

or,

$$g(\mu) = \frac{g_1}{1 + \frac{g_1}{2\pi} \ln(\frac{\mu^2}{\mu_1^2})}.$$

$g(\mu)$ decreases when μ increases. We also have:

$$\mu\frac{\partial g}{\partial \mu} = -\frac{g^2}{\pi} = \beta(g) < 0.$$

The theory is asymptotically free.

The minimum of the effective potential is obtained from:

$$\frac{\partial U_{eff}}{\partial \sigma} = \sigma_0 N\left[\frac{1}{g(\mu)} + \frac{1}{2\pi}\ln(\frac{\sigma_0^2}{\mu^2})\right] = 0,$$

which gives:

$$\sigma_0 = \mu e^{-\pi/g}.$$

We must shift the field σ in the lagrangian by this quantity:

$$\sigma = \sigma' + \sigma_0.$$

The lagrangian becomes:

$$\mathcal{L}' = \overline{\psi}^a i \gamma^\mu \partial_\mu \psi^a - \frac{N}{2g_0}(\sigma' + \sigma_0)^2 - \sigma_0 \overline{\psi}^a \psi^a - \sigma' \overline{\psi}^a \psi^a.$$

We notice the appearance of a mass term of fermions, with mass

$$M_F = \sigma_0 = \mu e^{-\pi/g}.$$

The theory was invariant at the beginning under the discrete chiral transformations, imposing masslessness of the fermions, but now, after renormalization and the shift to the ground state of the energy, the fermions have acquired a mass. This phenomenon is called dynamical mass generation and is due to the spontaneous breaking of the discrete chiral symmetry.

On the other hand, M_F is a physical quantity and should not depend on the particular choices of the arbitrary mass parameter μ. One verifies that it satisfies the equation

$$\frac{dM_F}{d\mu} = \frac{\partial M_F}{\partial \mu} + \frac{1}{\mu}\beta(g)\frac{\partial M_F}{\partial g} = 0.$$

M_F is renormalization group invariant.

Once it is fixed, the two other parameters, g and μ should disappear from the theory. Any choice of μ is compensated by a corresponding choice of g, to produce M_F.

At the beginning, the theory had a dimensionless parameter g. Now it has a dimensionful parameter, M_F, which is fixed by the physical conditions and sets the mass scale of the theory. There is no longer a free parameter in the theory. This phenomenon is called mass transmutation.

Summary

The Gross-Neveu model displays many interesting features of quantum field theory and illustrates in two dimensions several phenomena – asymptotic freedom, dynamical mass generation, mass transmutation – expected to occur in four-dimensional theories, which, however, are not completely or explicitly solved there.

Conclusion

The $1/N$ expansion method allows us to solve, at leading order of the expansion, theories and models nonperturbatively, probing directly dynamical phenomena which would not be reached in ordinary perturbation theory based on expansions with respect to the coupling constant.

This method is also applicable to QCD, but because of the presence of the gluon fields, which belong to the adjoint representation of the gauge symmetry group, its effects are less spectacular. Nevertheless, many simplifications occur and several qualitative features can be drawn about the properties of the theory.