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Integral equation for gauge invariant quark Green's function

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Objective

Investigate the possibilities of deriving integral or integro-differential equations for gauge invariant Green's functions. Those involve path-ordered gluon field phase factors. Here, we concentrate on two-point quark Green's functions, in which the path-ordered phase factor is made of a single straight line or more generally of a skew-polygonal line.

The starting point is a particular representation for the quark propagator in the presence of an external gluon field, where it is expressed as a series of terms involving path-ordered phase factors along successive straight lines. Then the corresponding quantized Green's function becomes expressed in terms of Wilson loops having skew-polygonal contours.

Definitions and conventions

Path-ordered gluon field phase factor along a line C_{yx} joining a point x to a point y, with an orientation defined from x to y:

$$U(C_{yx}; y, x) \equiv U(y, x) = Pe^{-ig \int_x^y dz^\mu A_\mu(z)}.$$

Parametrizing the line C with a parameter λ, $0 \leq \lambda \leq 1$, such that $x(0) = x$ and $x(1) = y$, a variation of C induces the following variation of U (Mandelstam, 1968):

$$\begin{aligned} \delta U(1,0) &= -ig\delta x^\alpha(1) A_\alpha(1) U(1,0) + igU(1,0) A_\alpha(0) \delta x^\alpha(0) \\ &\quad + ig \int_0^1 d\lambda U(1,\lambda) x'^\beta(\lambda) F_{\beta\alpha}(\lambda) \delta x^\alpha(\lambda) U(\lambda, 0), \end{aligned}$$

where $x' = \frac{\partial x}{\partial \lambda}$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$.

For paths defined along rigid lines, the variations inside the integral are related, with appropriate weight factors, to those of the end points.

Considering now a rigid straight line between x and y, a derivation at the end points yields:

$$\frac{\partial U(y,x)}{\partial y^\alpha} = -igA_\alpha(y)U(y,x) + ig(y-x)^\beta \int_0^1 d\lambda\, \lambda\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\partial U(y,x)}{\partial x^\alpha} = +igU(y,x)A_\alpha(x) + ig(y-x)^\beta \int_0^1 d\lambda\, (1-\lambda)\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Conventions to represent the contributions of the integrals:

$$\frac{\bar{\delta} U(y,x)}{\bar{\delta} y^{\alpha+}} \equiv ig(y-x)^\beta \int_0^1 d\lambda\, \lambda\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$
$$\frac{\bar{\delta} U(y,x)}{\bar{\delta} x^{\alpha-}} \equiv ig(y-x)^\beta \int_0^1 d\lambda\, (1-\lambda)\, U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Wilson loop

$$\Phi(C) = \frac{1}{N_c} \mathrm{tr} P e^{-ig \oint_C dx^\mu A_\mu(x)}.$$

Vacuum expectation value:

$$W(C) = \langle \Phi(C) \rangle.$$

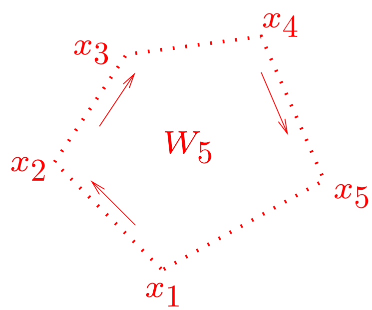
Functional representation:

$$W(C) = e^{F(C)}.$$

In perturbation theory, $F(C)$ is given by the sum of all connected diagrams, the connection being defined with respect to the contour C (Dotsenko and Vergeles, 1980). For large contours and large N_c, $F(C)$ is proportional to the minimal surface with contour C (Makeenko and Migdal, 1980).

If the contour C is a skew-polygon C_n with n sides and n successive marked points x_1, x_2, …, x_n at the cusps, then we write:

$$W(x_n, x_{n-1}, \ldots, x_1) = W_n = e^{F_n(x_n, x_{n-1}, \ldots, x_1)} = e^{F_n}.$$



$$W_5(x_5, x_4, \ldots, x_1) = e^{F_5(x_5, \ldots, x_1)}$$

Two-point Green's functions

The gauge invariant two-point quark Green's function is defined as

$$S_{\alpha\beta}(x,x';C_{x'x}) = -\frac{1}{N_c}\,\langle\overline{\psi}_\beta(x')\,U(C_{x'x};x',x)\,\psi_\alpha(x)\rangle.$$

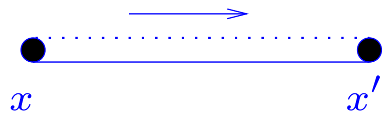
For skew-polygonal lines with n sides and $n-1$ junction points y_1, y_2, $\ldots$,y_{n-1} between the segments, we define:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = -\frac{1}{N_c}\,\langle\overline{\psi}(x')U(x',y_{n-1})U(y_{n-1},y_{n-2})\ldots U(y_1,x)\psi(x)\rangle.$$

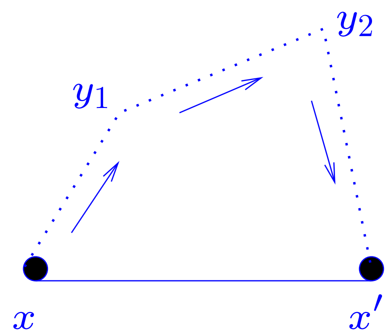
For one straight line, one has:

$$S_{(1)}(x,x') \equiv S(x,x') = -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,U(x',x)\,\psi(x)\rangle.$$

Pictorially:



$$S(x,x') \equiv S_{(1)}(x,x') = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',x)\, \psi(x) >$$



$$S_{(3)}(x,x';y_2,y_1) = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',y_2)U(y_2,y_1)U(y_1,x)\, \psi(x) >$$

Quark propagator in the external gluon field

A two-step quantization. One first integrates with respect to the quark fields. This produces in various terms the quark propagator in the presence of the gluon field. Then one integrates with respect to the gluon field through Wilson loops.

To make Wilson loops appear, one needs an appropriate representation for the quark propagator in extenal field. We use a representation which involves phase factors along straight lines together with the full quark Green's function $S_{(1)} \equiv S$ (F. Jugeau and H.S., 2003). Generalization of a representation introduced by Eichten and Feinberg, 1981, for heavy quarks.

The quark propagator in the external gluon field A, designated by $S(x,x';A)$, satisfies the usual equation

$$\Big(i\gamma.\partial_{(x)} - m - g\gamma.A(x)\Big)S(x,x';A) = i\delta^4(x-x').$$

The starting point of the representation is the gauge covariant composite object $\widetilde{S}_0(x,x')$, made of a free fermion propagator $S_0(x,x')$ (without color group content) multiplied by the path-ordered phase factor $U(x,x')$ taken along the straight line xx':

$$\Big[\widetilde{S}_0(x,x')\Big]^a_{\ b} \equiv S_0(x,x')\Big[U(x,x')\Big]^a_{\ b}.$$

[a,b: color indices.]

$\widetilde{S}_0$ satisfies the following equation with respect to x:

$$\left(i\gamma.\partial_{(x)} - m - g\gamma.A(x)\right)\widetilde{S}_0(x,x') = i\delta^4(x-x') + i\gamma^\alpha \frac{\bar{\delta}U(x,x')}{\bar{\delta}x^{\alpha+}} S_0(x,x').$$

The quantity $-i(i\gamma.\partial_{(x)} - m - g\gamma.A(x))\delta^4(x-x')$ is the inverse of the quark propagator $S(x,x';A)$ in the presence of the external gluon field A. Reversing the equation with respect to $S(A)^{-1}$, one obtains an equation for $S(A)$ in terms of $\widetilde{S}_0$:

$$S(x,x';A) = \widetilde{S}_0(x,x') + \int d^4x'' \frac{\bar{\delta}\widetilde{S}_0(x,x'')}{\bar{\delta}x''^{\alpha-}} \gamma^\alpha S(x'',x';A).$$

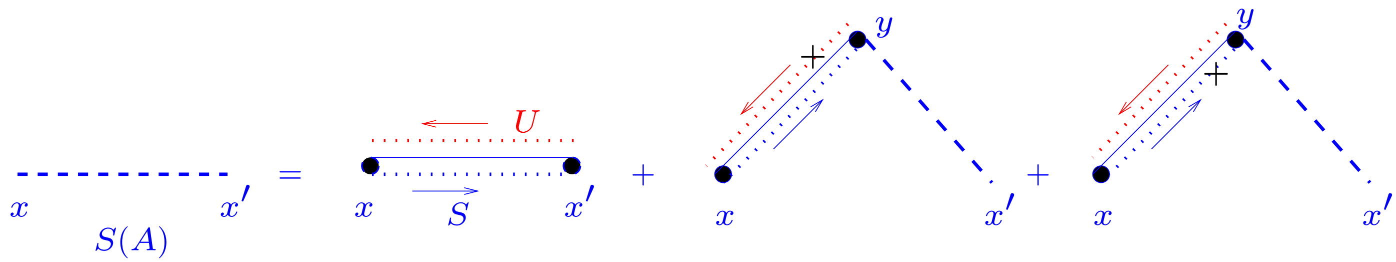
In order to sum self-energy effects, one can use for the expansion of the propagator $S(A)$, instead of the free propagator S_0, the full gauge invariant Green's function S. We define a generalized version of the gauge covariant object $\widetilde{S}_0$ by replacing in it S_0 with S:

$$\left[\widetilde{S}(x,x')\right]^a_{\ b} \equiv S(x,x')\left[U(x,x')\right]^a_{\ b}.$$

Proceeding as before one arrives at the expansion of $S(A)$ around S.

$$S(x,x';A) = S(x,x')U(x,x') + \left(S(x,y)\frac{\bar{\delta}U(x,y)}{\bar{\delta}y^{\alpha -}} + \frac{\bar{\delta}S(x,y)}{\bar{\delta}y^{\alpha +}}U(x,y) \right) \gamma^{\alpha} S(y,x';A).$$

Pictorially:



This yields an expansion of $S(A)$ in terms of the gauge invariant Green's function S and explicit phase factors along straight lines.

Functional relations for Green's functions

Systematic use of the expansion of the quark propagator in external field.

Consider the Green's function $S_{(n)}$:

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = -\frac{1}{N_c}\langle\overline{\psi}(x')U(x',y_{n-1})U(y_{n-1},y_{n-2})\ldots U(y_1,x)\psi(x)\rangle.$$

Integrate with respect to the quark fields:

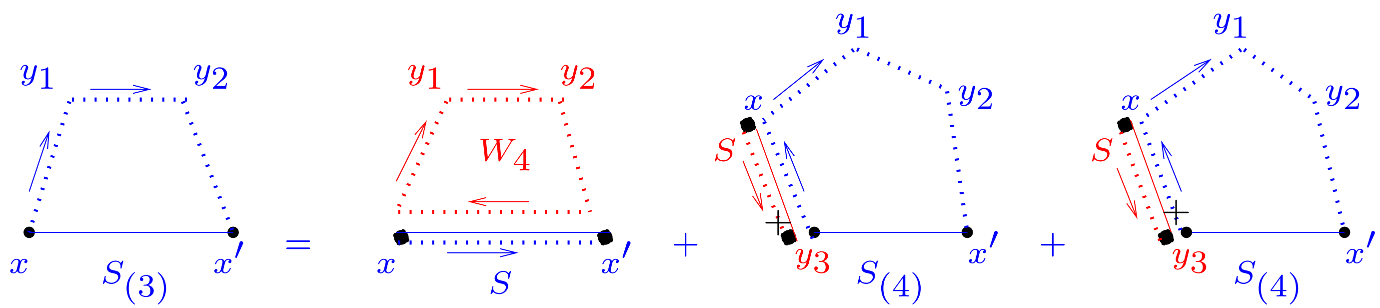
$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = \frac{1}{N_c}\langle U(x',y_{n-1})U(y_{n-1},y_{n-2})\cdots U(y_1,x)S(x,x';A)\rangle.$$

Use the expansion found for $S(A)$:

$$S(x,x';A) = S(x,x')U(x,x') + \left(S(x,y)\frac{\bar{\delta}U(x,y)}{\bar{\delta}y^{\alpha-}} + \frac{\bar{\delta}S(x,y)}{\bar{\delta}y^{\alpha+}}U(x,y)\right)\gamma^{\alpha}S(y,x';A).$$

$$
\begin{aligned}
S_{(n)}(x,x';y_{n-1},\ldots,y_1) &= S(x,x')\, e^{F_{n+1}(x',y_{n-1},\ldots,y_1,x)} \\
&+ \left(\frac{\bar{\delta} S(x,y_n)}{\bar{\delta} y_n^{\alpha+}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta} y_n^{\alpha-}}\right)\gamma^{\alpha}\, S_{(n+1)}(y_n,x';y_{n-1},\ldots,y_1,x).
\end{aligned}
$$

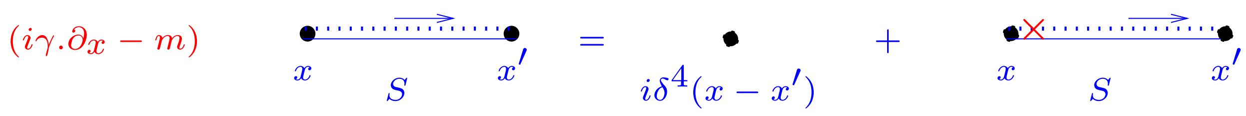
Graphical representation for $n=3$:

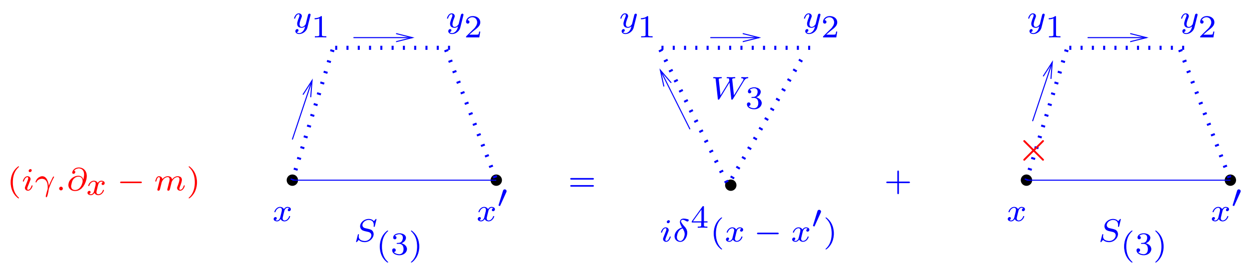


Equations of motion

$$(i\gamma.\partial_{(x)} - m)S_{(n)}(x,x';y_{n-1},\ldots,y_1) = i\delta^4(x-x')e^{F_n(x,y_{n-1},\ldots,y_1)}$$
$$+i\gamma^{\mu}\frac{\bar{\delta}S_{(n)}(x,x';y_{n-1},\ldots,y_1)}{\bar{\delta}x^{\mu-}}.$$

Graphical representation of this equation for $n=1$ and $n=3$:





Integral equation

$\bar{\delta}S/\bar{\delta}x^{\mu-}$ and $\bar{\delta}S_{(n)}/\bar{\delta}x^{\mu-}$ can be expressed, with the aid of the functional relations, in terms of Wilson loop derivatives and Green's functions.

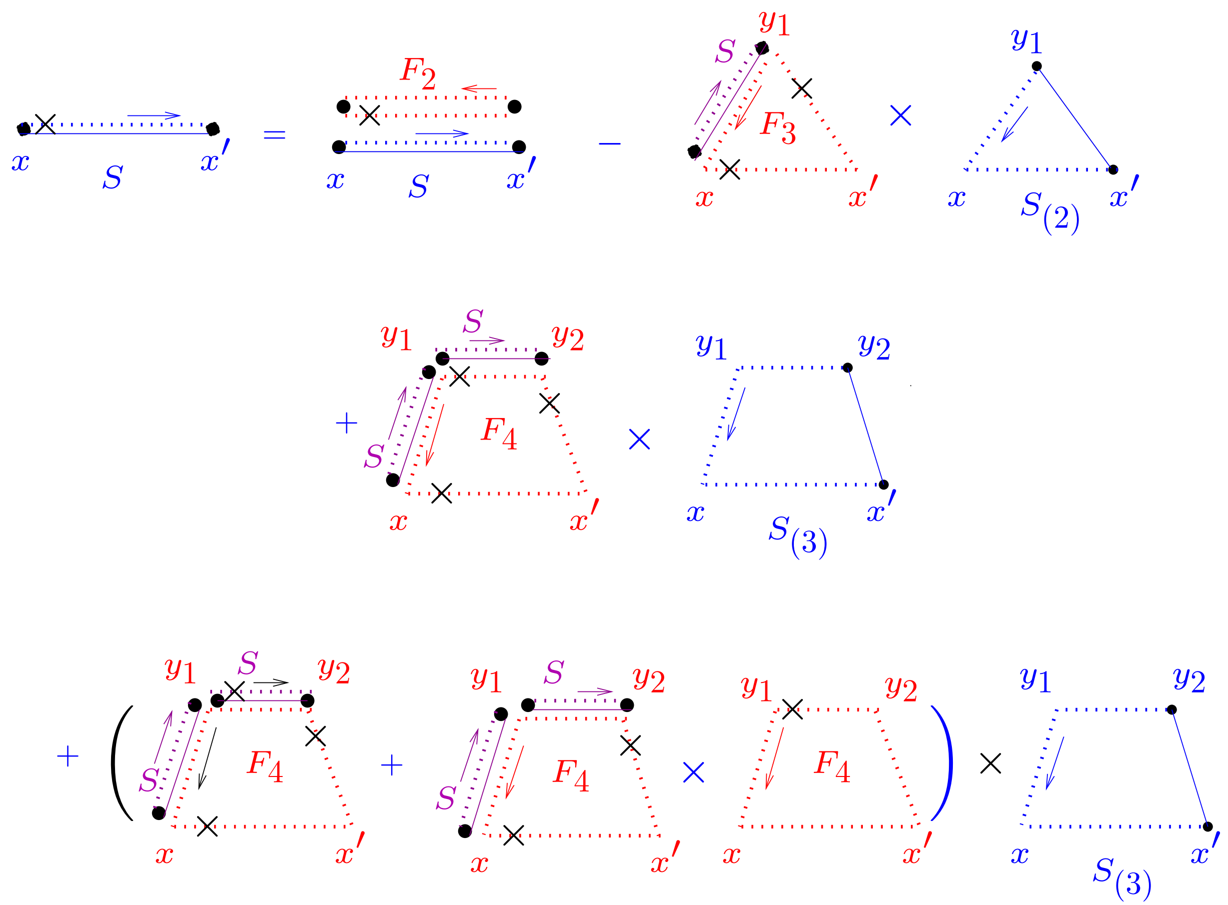
$$\frac{\bar{\delta}S_{(n)}}{\bar{\delta}x^{\mu-}} = \frac{\bar{\delta}F_{n+1}}{\bar{\delta}x^{\mu-}}\, S_{(n)} + \Big(\frac{\bar{\delta}}{\bar{\delta}x^{\mu-}} - \frac{\bar{\delta}F_{n+1}}{\bar{\delta}x^{\mu-}}\Big)\,\Big(\frac{\bar{\delta}S(x,y_n)}{\bar{\delta}y_n^{\alpha+}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta}y_n^{\alpha-}}\Big)\,\gamma^{\alpha}\, S_{(n+1)}.$$

At the end, one obtains for $\bar{\delta}S/\bar{\delta}x^{\mu-}$ a series expansion in terms of the Green's functions $S_{(n)}$, each term involving a kernel expressed in terms of Wilson loop derivatives and Green's function S and its derivative.

$$\frac{\bar{\delta} S(x,x')}{\bar{\delta} x^{\mu -}} = K_{1\mu -}(x',x)\, S(x,x') + K_{2\mu -}(x',x,y_1)\, S_{(2)}(y_1,x';x)$$
$$+ \sum_{n=3}^{\infty} K_{n\mu -}(x',x,y_1,\ldots,y_{n-1})\, S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2}).$$

The kernel K_n contains globally n derivatives of Wilson loops and also the Green's function S and its derivative.

Graphical representation up to third-order terms:



At short-distances, governed by perturbation theory, each derivation introduces a new power of the coupling constant and therefore the dominant terms in the expansion are the lowest-order ones. At large-distances, Wilson loops are saturated by the minimal surfaces having as supports the contours. Here also, the dominant contributions come from the lowest-order derivative terms. Therefore the expansion above can be considered in general as a perturbative one.
Thus the dominant part of the kernel comes from the second-order term (the first-order one being zero for symmetry reasons).

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu \frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}.$$

$$\frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}} \simeq -\int d^4y_1 \, \frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta}x^{\mu-}\bar{\delta}y_1^{\alpha_1+}} \, e^{F_3(x',x,y_1)} \, S(x,y_1)\, \gamma^{\alpha_1} \, S(y_1,x').$$

Analyticity properties of the Green's function

One of the advantages of paths along straight lines is the fact that the expressions of the corresponding Green's functions become dependent only on the end points of the paths. Simple transition to momentum space by Fourier transformation. Much of the informations on Green's functions are provided from momentum space, since it is there that their spectral properties are determined.

However, the quark two-point gauge invariant Green's functions hold a particular position. Because of confinement of colored objects, it is not possible to cut the path joining the quark to the antiquark by inserting in it a complete set of physical states, which are color singlets. This feature seems to suggest that gauge invariant two-point Green's functions should not have any singularities.

The situation is, however, more complex. Gauge invariant two-point Green's functions possess singularities originated from perturbation theory. In the integral equation it is the presence of the free quark propagator which generates the singularities of the complete solution. An analysis, starting from perturbation theory, is therefore necessary.

We admit that, in a domain where perturbation theory is valid, it is meaningful to consider quarks and gluons as physical particles with positive energies, described by corresponding physical states. It is then advantageous to consider the path-ordered phase factor U in its representation given by the series expansion in terms of the coupling constant g, the nth-order term of the expansion containing $(n-1)$ gluon fields ($n \geq 1$).

$$
\begin{aligned}
S(x,x') &= -\frac{1}{N_c}\,\langle\overline{\psi}(x')\,U(x',x)\,\psi(x)\rangle \\
&= -\frac{1}{N_c}\left\{\langle\overline{\psi}(x')\left[1 - ig\int_{C_{x'x}} dz_1^{\alpha_1} A_{\alpha_1}(z_1) + \right.\right. \\
&\quad + \sum_{n=2}^{\infty}(-ig)^n \int_{C_{x'x}} \cdots \int_{C_{x'x}} dz_1^{\alpha_1} \cdots dz_n^{\alpha_n}\,\theta_C(x', z_n, \ldots, z_1, x) \\
&\quad \left.\left. \times A_{\alpha_n}(z_n) \cdots A_{\alpha_1}(z_1)\right]\psi(x)\rangle\right\}.
\end{aligned}
$$

In operator formalism, the above Green's function involves two kinds of orderings for its defining fields. The first is the path-ordering (or P-ordering) which concerns the color index arrangements of the gluon fields according to their positions on the path. The second is the time-ordering (T-ordering) or chronological product which enters in the definitions of Green's functions and operates once the P-ordering is done.

Once the timelike or spacelike nature of the distance between the quark and the antiquark is fixed, the nature of the mutual distances of the gluon fields in the Green's function S is also fixed in the same way, because of their alignement along the segment joining the quark to the antiquark. Therefore, the chronological product of the nth-order terms in S reduces to two terms, defined by the relative time between the quark and the antiquark. For timelike $(x' - x)$, if $({x'}^0 - x^0) > 0$ then the T-ordering will coincide with the P-ordering, while if $({x'}^0 - x^0) < 0$ the T-ordering will be the opposite of the P-ordering (with a change of sign for the fermion fields), the color indices being already fixed from the P-ordering. We are in a situation which is very similar to the case of the ordinary two-point function, with the difference that for an nth-order term there are $(n+1)$ fields instead of two ($(n-1)$ gluon, one quark and one antiquark fields).

Using for each of the two products which make the T-product the spectral analysis with intermediate states, taking into account the bounds on the parameters of the P-ordering and using causality, one arrives at a generalized form of the Källén–Lehmann representation for the Green's function S in momentum space, in which the cut starts on the real axis from the quark mass squared m^2 and extends to infinity. The generalization is due to the fact that each gluon field is integrated along the path and this introduces, when using for the latter a dimensionless parameter λ varying between 0 and 1, a multiplicative factor $(x'-x)$, which is converted in momentum space into a derivation operator; each such factor increases by one unit the power of the denominator of the dispersion integral.

Fourier transform of the Green's function $S(x,x')$, taking also into account translation invariance:

$$S(x,x') = S(x-x') = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip.(x-x')}\, S(p).$$

$S(p)$ has the following representation in terms of real spectral functions $\rho_1^{(n)}$ and $\rho_0^{(n)}$ $(n = 1,\ldots,\infty)$:

$$S(p) = i\int_0^{\infty} ds' \sum_{n=1}^{\infty} \frac{\left[\,\gamma.p\,\rho_1^{(n)}(s') + \rho_0^{(n)}(s')\,\right]}{(p^2 - s' + i\varepsilon)^n}.$$

We assume that the above representation, obtained from the domain of perturbation theory, remains also valid in non-perturbative regimes. One expects that the resulting singularities are strong enough to screen the quark pole and other physical type singularities.