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Quarkonium bound state equation in the Wilson loop approach with minimal surfaces

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Approach based on the use of the path-ordered phase factor along a line C joining a point x to a point y (link):

$$U(C_{yx}; y, x) \equiv U(y, x) = P e^{-ig \int_x^y dz^\mu A_\mu(z)},$$

where $A_\mu = \sum_a A_\mu^a t^a$, A_μ^a ($a = 1, \ldots, N_c^2 - 1$) are the gluon fields and t^a the generators of the color gauge group $SU(N_c)$ in the fundamental representation. U is a covariant object under gauge transformations:

$$U(C; y, x) \longrightarrow \Omega(y) U(C; y, x) \Omega^{-1}(x).$$

Together with the quark fields, the object

$\overline{\psi}(y) U(C; y, x) \psi(x)$ is gauge invariant.

Another gauge invariant object is obtained by considering the trace of U in color space along a closed contour C:

$$\Phi(C) = \frac{1}{N_c} \mathrm{tr}_c P e^{-ig \oint_C dx^\mu A_\mu(x)} .$$

This defines the Wilson loop.
Its vacuum expectation value is denoted $W(C)$:

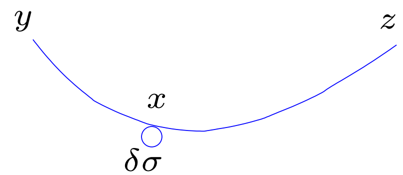
$$W(C) = \langle \Phi(C) \rangle_A .$$

Wilson (1974) showed that in the static limit $W(C)$ is a useful tool for devising a criterion for the confinement of quarks (area law).

Try to deduce the large-distance properties of the theory from the properties of the Wilson loop.

Nambu (1979), Polyakov (1979), Makeenko and Migdal (1979).

Mandelstam's relation:



$$\frac{\delta U(y,z)}{\delta\sigma^{\alpha\beta}(x)} = -igU(y,x)F_{\alpha\beta}(x)U(x,z),$$

where F is the field strength, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. Derive with respect to x (∇ is the covariant derivative):

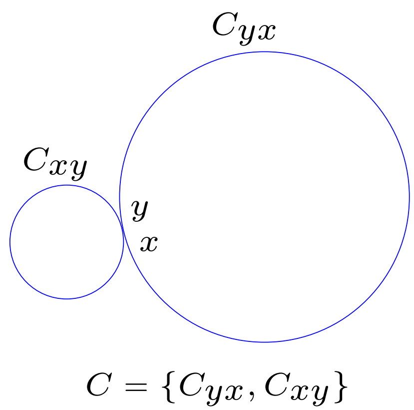
$$\frac{\partial}{\partial x^\mu}\frac{\delta U(y,z)}{\delta\sigma^{\alpha\beta}(x)} = -igU(y,x)(\nabla_\mu F_{\alpha\beta}(x))U(x,y).$$

$$\frac{\partial}{\partial x^\mu}\frac{\delta U(y,z)}{\delta\sigma^{\alpha\beta}(x)} = -igU(y,x)(\nabla_\mu F_{\alpha\beta}(x))U(x,y).$$

Specialize to closed contours C (Wilson loop averages).
Cyclic permutations of the indices (μ,α,β): The Bianchi identity. $\Longrightarrow$

$$\varepsilon^{\nu\mu\alpha\beta}\frac{\partial}{\partial x^\mu}\frac{\delta W(C)}{\delta\sigma^{\alpha\beta}(x)} = 0.$$

Contraction of μ and α: Equation of motion of the gluon field (in the large-N_c limit internal quark contributions can be neglected). $\nabla^\alpha F_{\alpha\beta}(x) \sim \delta/\delta A^\beta(x)$. Only points that are at x can contribute. Apart from x itself, these are the self-intersecting points on C.



In the large-N_c limit:

$$\frac{\partial}{\partial x^\alpha}\frac{\delta W(C)}{\delta\sigma_{\alpha\beta}} = -i\frac{g^2 N_c}{2}\oint_C dy^\beta \delta^4(y-x) W(C_{yx}) W(C_{xy}).$$

Loop equations or Makeenko-Migdal equations.

Equations solved in two dimensions by Kazakov and Kostov (1980).

Solutions in terms of the areas delimited by the closed contours.

Renormalizability: Dotsenko and Vergeles (1980) and Brandt et al. (1981).

For large contours and at large N_c, Makeenko and Migdal showed that minimal surfaces are asymptotic solutions to the loop equations. They also satisfy in general (for simple enough contours) the factorization property.

Our starting point is the idea that minimal surfaces can represent, for large distances and at large N_c, solutions to the Wilson loop averages.

$$W(C) = e^{-i\sigma A(C)}.$$

One can show the following properties.

1) Among various types of surface, having as support the contour C, only the minimal surface satisfies the Bianchi identity.

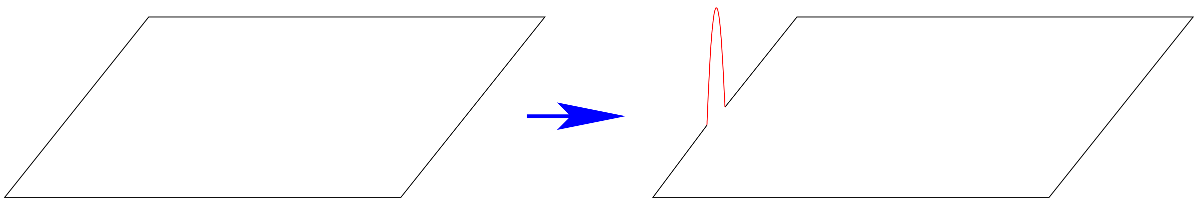
2) The loop equation is satisfied by the minimal surface provided one defines the unrenormalized coupling constant g through the following relation with the string tension σ:

$$g^2 N_c = \lim_{a \to 0} C \sigma a^2 .$$

But in the presence of short-distance effects, g is defined in terms of Λ_{QCD} and vanishes logarithmically with the regulator. In that case, the minimal surface represents only a partial contribution to the Wilson loop average and it is the junction between large and short distances that should determine the relation between σ and Λ_{QCD}. Problem not yet solved exactly.
We consider henceforth only large-distance contributions.

Some mathematical properties of minimal surfaces

Any local deformation of the contour modifies the minimal surface in its internal part.



The new minimal surface is no longer plane.

The deformations inside the minimal surface can be calculated in terms of the deformations of the contour with the aid of the Green function of the defining equation of the minimal surface (Lüscher, Symanzik, Weisz, 1980).

Problem exactly solved when the initial or background minimal surface is plane. In general, rigid deformations of the contour along finite parts can be calculated as superpositions of local deformations.

Quark-antiquark bound states

$$G \equiv \langle \overline{\psi}_2(x_2)U(x_2,x_1)\psi_1(x_1)\,\overline{\psi}_1(x_1')U(x_1',x_2')\psi_2(x_2')\rangle_{A,q_1,q_2}.$$

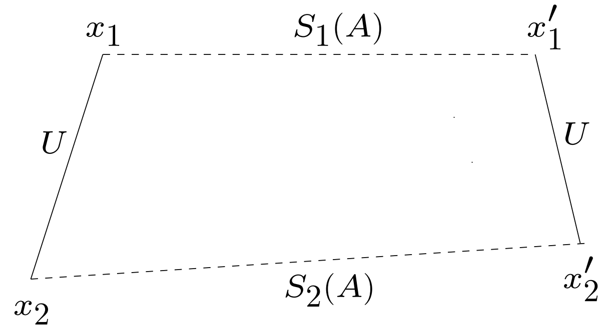
U taken along straight lines.

Integrate with respect to the quark fields (large-N_c).

$$G = -\langle \mathrm{tr}_c\, U(x_2,x_1)S_1(A;x_1,x_1')U(x_1',x_2')S_2(A;x_2',x_2)\rangle_A.$$

$S(A)$ is the quark propagator in the presence of the external gluon field A.

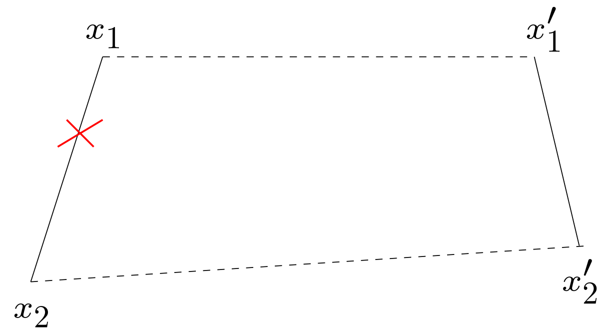
$$\Big(i\gamma.\partial_{(x)} - m - g\gamma.A(x)\Big)S(A;x,x') = i\delta^4(x-x').$$


x_1
$S_1(A)$
x_1'
U
U
x_2
$S_2(A)$
x_2'

Action of the Dirac operators on G:

$$(i\gamma.\partial_{(x_1)} - m_1)G = -i\langle \mathrm{tr}_c\, U\delta^4(x_1 - x_1')US_2\rangle_A$$
$$-i\gamma^\alpha\langle \mathrm{tr}_c \int_0^1 d\lambda(1-\lambda)\frac{\delta U(x_2,x_1)}{\delta x^\alpha(\lambda)}S_1US_2\rangle_A,$$

$$G(-i\gamma.\overleftarrow{\partial}_{(x_2)} - m_2) = -i\langle \mathrm{tr}_c\, US_1U\delta^4(x_2' - x_2)\rangle_A$$
$$+i\langle \mathrm{tr}_c \int_0^1 d\lambda\lambda\frac{\delta U(x_2,x_1)}{\delta x^\beta(\lambda)}S_1US_2\rangle_A\gamma^\beta.$$



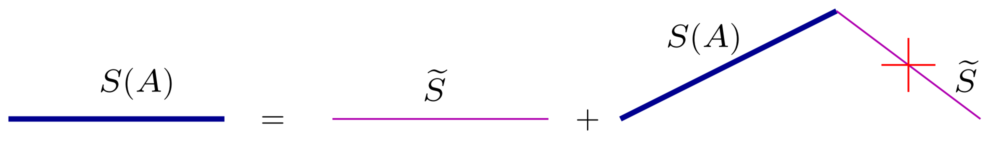
Representation for $S(A)$

Expand $S(A)$ around the free propagator with insertions of phase factors U.

$$\left[\widetilde{S}(x,x')\right]^a_{\ b} \equiv S_0(x-x')\left[U(x,x')\right]^a_{\ b}.$$

$S(A)$ then satisfies the following integral equation, which can be converted into an iteration series:

$$S(x,x') = \widetilde{S}(x,x') - \int d^4x'' S(x,x'')\gamma^\alpha \int_0^1 d\lambda\, \lambda \frac{\delta}{\delta x^\alpha(\lambda)} \widetilde{S}(x'',x').$$



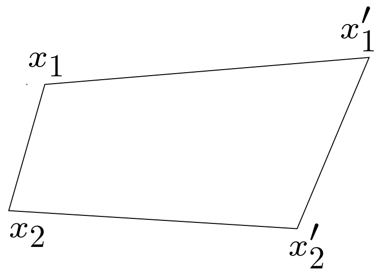
(Generalization of the static case; Eichten and Feinberg, 1981.)

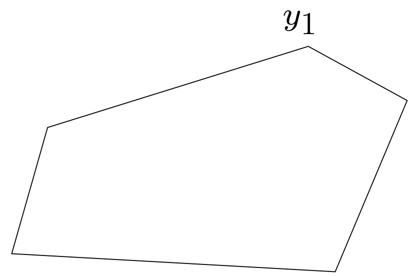
This leads to a series expansion of G in terms of Wilson loops.

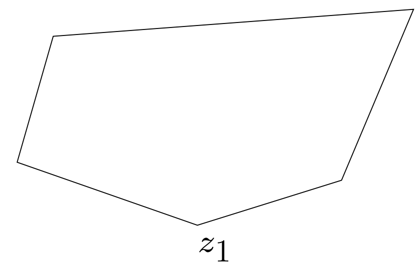
$$G = \sum_{i,j=1}^{\infty} G_{i,j},$$

where $G_{i,j}$ can be expressed in terms of a Wilson loop along a skew-polygonal contour with $(i+1)+(j+1)$ vertices.

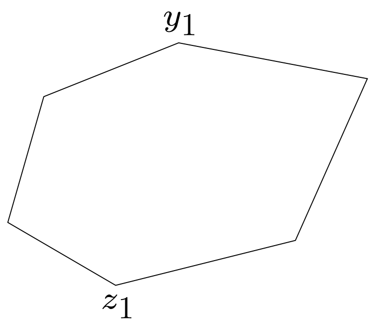
$G =$

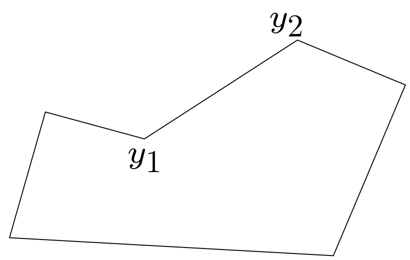


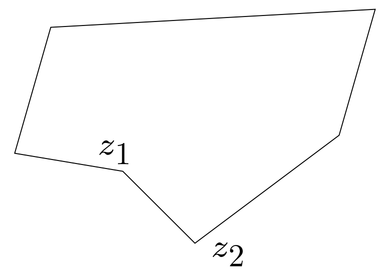




$G_{1,1} \quad + \quad G_{2,1} \quad + \quad G_{1,2}$







$+ \quad G_{2,2} \quad + \quad G_{3,1} \quad + \quad G_{1,3}$

$+ \cdots \quad G_{i,j} \quad + \cdots .$

The two Dirac-type equations become now:

$$(i\gamma.\partial_{(x_1)} - m_1)G = -i\delta^4(x_1 - x_1')\sum_{j=1}^{\infty} G_{0,j}$$

$$+i\gamma^{\alpha}\sum_{i,j=1}^{\infty}\int_0^1 d\lambda(1-\lambda)\frac{\delta}{\delta x^{\alpha}(\lambda)}G_{i,j}\Big|_{x(\lambda)\in x_1x_2},$$

$$G(-i\gamma.\overleftarrow{\partial}_{(x_2)} - m_2) = +i\delta^4(x_2 - x_2')\sum_{i=1}^{\infty} G_{i,0}$$

$$-i\sum_{i,j=1}^{\infty}\int_0^1 d\lambda\lambda\frac{\delta}{\delta x^{\beta}(\lambda)}G_{i,j}\gamma^{\beta}\Big|_{x(\lambda)\in x_1x_2}.$$

The two equations are compatible (integrable); mainly due to the following property (which depends on the Bianchi identity):

$$\left(\frac{\delta}{\delta x^{\beta}(\lambda')}\frac{\delta}{\delta x^{\alpha}(\lambda)}-\frac{\delta}{\delta x^{\alpha}(\lambda)}\frac{\delta}{\delta x^{\beta}(\lambda')}\right)A(C)=0.$$

$\Longrightarrow$ The relative time between x_1 and x_2 should not play any dynamical role, its evolution law being determined by the difference of the two equations.

Each $G_{i,j}$ contains $(i+j)$ functional derivatives of the Wilson loop. The action of a new derivative along the segment x_1x_2 gives rise to different categories of derivatives which can be grouped according to their connectedness with respect to the minimal surface $A_{i,j}$.

For example, $G_{2,1}$ is proportional to $e^{-i\sigma A_{2,1}}$, but contains one derivative along the segment y_1x_1':

$$G_{2,1} \sim \frac{\delta}{\delta y} e^{-i\sigma A_{2,1}} \Big|_{y \in y_1 x_1'} = -i\sigma \frac{\delta A_{2,1}}{\delta y} e^{-i\sigma A_{2,1}}.$$

The new derivative along x_1x_2 yields:

$$\begin{aligned} \frac{\delta}{\delta x} G_{2,1} \quad &\sim \quad \left[(-i\sigma)^2 \frac{\delta A_{2,1}}{\delta x}\frac{\delta A_{2,1}}{\delta y} - i\sigma \frac{\delta^2 A_{2,1}}{\delta x \delta y}\right] e^{-i\sigma A_{2,1}} \\ &= \quad -i\sigma \frac{\delta A_{2,1}}{\delta x} G_{2,1} - i\sigma \frac{\delta^2 A_{2,1}}{\delta x \delta y} e^{-i\sigma A_{2,1}}. \end{aligned}$$

We can thus group terms according to the number of derivatives of A they contain (those containing necessarily $\delta/\delta x$; those not containing $\delta/\delta x$ are parts of the definitions of $G_{i,j}$).

We consider here in detail the case of terms proportional to one derivative. (The other cases can be treated in a similar way.) One thus has:

$$\frac{\delta}{\delta x(\lambda)} G_{i,j} \bigg|_{x(\lambda)\in x_1 x_2} = -i\sigma \frac{\delta A_{i,j}}{\delta x} G_{i,j} + \cdots \; .$$

$\Longrightarrow$

$$\frac{\delta}{\delta x(\lambda)} G \bigg|_{x(\lambda)\in x_1 x_2} = \sum_{i,j} \frac{\delta}{\delta x} G_{i,j} = \sum_{i,j} -i\sigma \frac{\delta A_{i,j}}{\delta x} G_{i,j} + \cdots \; .$$

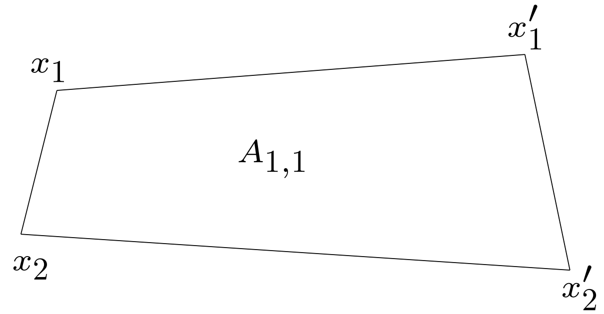
Replacing the above result in the right-hand side of the Dirac-type equation for quark 1, one obtains:

$$(i\gamma.\partial_{(x_1)} - m_1)G = -i\delta^4(x_1 - x_1')\sum_{j=1}^{\infty} G_{0,j}$$

$$+i\gamma^\alpha \sum_{i,j=1}^{\infty} \int_0^1 d\lambda (1-\lambda)(-i\sigma)\frac{\delta A_{i,j}}{\delta x^\alpha(\lambda)}\bigg|_{x(\lambda)\in x_1 x_2} G_{i,j} + \cdots \ .$$

In order to have a bound state, it is necessary that both sides of the equation have the same pole; the left-hand side involves G; the right-hand side involves a series with $G_{i,j}$ each with a different coefficient. It necessary that the series sums up coherently to produce a term containing G. (Each $G_{i,j}$ does not have a pole, containing a finite number of free quark propagators.)

All minimal surfaces $A_{i,j}$ have two common fixed segments, $x_1 x_2$ and $x'_1 x'_2$. All $A_{i,j}$ for $i > 1$ or $j > 1$ are fluctuating surfaces around the fixed surface $A_{1,1}$:



Therefore, each derivative $\delta A_{i,j}/\delta x$ can be expanded around the fixed derivative $\delta A_{1,1}/\delta x$, their differences representing fluctuation effects:

$$\frac{\delta A_{i,j}}{\delta x} = \frac{\delta A_{1,1}}{\delta x} + \text{ fluctuations.}$$

Replacing the above expansion in the Dirac-type equation, one obtains:

$$(i\gamma.\partial_{(x_1)} - m_1)G = -i\delta^4(x_1 - x_1')\sum_{j=1}^{\infty} G_{0,j}$$
$$+i\gamma^\alpha \sum_{i,j=1}^{\infty}\int_0^1 d\lambda(1-\lambda)(-i\sigma)\frac{\delta A_{1,1}}{\delta x^\alpha(\lambda)}\bigg|_{x(\lambda)\in x_1x_2} G_{i,j} + \text{fluctuations} + \cdots \ ,$$

$$(i\gamma.\partial_{(x_1)} - m_1)G = -i\delta^4(x_1 - x_1')\sum_{j=1}^{\infty} G_{0,j}$$
$$+\sigma\gamma^\alpha \int_0^1 d\lambda(1-\lambda)\frac{\delta A_{1,1}}{\delta x^\alpha(\lambda)}\bigg|_{x(\lambda)\in x_1x_2} G + \text{fluctuations} + \cdots \ .$$

Fluctuation terms add up incoherently; they cannot contribute to pole terms. One obtains the bound state equation:

$$(i\gamma.\partial_{(x_1)} - m_1)\Phi = \sigma\gamma^\alpha \int_0^1 d\lambda(1-\lambda)\frac{\delta A_{1,1}}{\delta x^\alpha(\lambda)}\bigg|_{x(\lambda)\in x_1x_2}\Phi,$$

$$\Phi = -i < 0|\overline{\psi}_2(x_2)U(x_2,x_1)\psi_1(x_1)|P > .$$

Plus additional contributions to the kernel of the equation coming from higher-order derivatives of minimal surfaces involving always $\delta/\delta x$ along x_1x_2. These will not be considered here.

A similar equation is obtained with the Dirac operator of the antiquark.

$\delta A_{1,1}/\delta x$ is calculated in the limit of large time separation between x_1x_2 and $x_1'x_2'$, which defines the bound state limit.

After eliminating the relative time in the cm frame, one obtains a reduced equation of the Breit-Salpeter type:

$$\Big[P_0 - (h_{10} + h_{20}) - \gamma_{10}\gamma_1^{\mu} A_{1\mu} - \gamma_{20}\gamma_2^{\mu} A_{2\mu}\Big]\psi = 0,$$

where h_{10} and h_{20} are the free Dirac Hamiltonians and the potentials A are defined as

$$A_{1\mu} = \sigma \int_0^1 d\lambda (1-\lambda) \frac{\delta A_{1,1}}{\delta x^{\mu}(\lambda)}, \qquad A_{2\mu} = \sigma \int_0^1 d\lambda\, \lambda \frac{\delta A_{1,1}}{\delta x^{\mu}(\lambda)}.$$

The functional derivative $\delta/\delta x(\lambda)$ acts on the surface $A_{1,1}$ as

$$\frac{\delta A_{1,1}}{\delta x^{\alpha}(\lambda)} = -\sqrt{-x'^2}\frac{\dot{x}^{t\alpha}(\lambda)}{\sqrt{\dot{x}^{t2}(\lambda)}}.$$

$x' = x_2 - x_1$; $\dot{x}^{t\alpha}/\sqrt{\dot{x}^{t2}}$ is the slope of the surface in the orthogonal direction to x' at $x(\lambda)$.

One associates this slope with the momenta of the quarks.

The interaction potential, which is vector-like, is relativistic and valid for any kind of quark masses.

It is a nonlocal function of the quark three-momenta (in the cm frame).

It contains, as a leading part, the linear confining potential σr.

The contributions of the relativistic efects can be best understood by considering particular limits.

One-particle limit

One of the quarks is infinitely massive.

$$\left[p_{10} - h_{10} - A_0 + \gamma_{10}\boldsymbol{\gamma}.\mathbf{A}\right]\psi = 0.$$

Taking further the limit of a heavy quark for the remaining quark, one has for the potentials to order $1/c^2$:

$$A_0 = \sigma r\left(1 + \frac{\mathbf{L}^2}{6m_1^2 r^2}\right), \qquad \mathbf{A} = \frac{\sigma r}{3}\frac{\mathbf{p}^t}{m_1}.$$

($r = |\mathbf{x}| = |\mathbf{x}_1 - \mathbf{x}_2|$; $\mathbf{L}$: orbital angular momentum of the quark; $\mathbf{p}^t$: quark momentum orthogonal to $\mathbf{x}$.)

Interpretation: The interaction potentials are represented by the energy-momentum vector of the straight-segment joining the two quarks and carrying a linear energy density σ.

σr : the static energy of the straight-segment.

$\sigma r\mathbf{L}^2/(6m_1^2r^2)$: its rotational energy.

$\sigma r\mathbf{p}^t/(3m_1)$: its rotational momentum.

The corresponding Hamiltonian, in two-component spinor-space, is:

$$\begin{aligned} H &= \frac{\mathbf{p}^2}{2m_1} + \sigma r - \frac{2\hbar\sigma}{\pi m_1} - \frac{(\mathbf{p}^2)^2}{8m_1^3} + \frac{\hbar^2}{4m_1^2}\frac{\sigma}{r} \\ &\quad -\frac{\sigma}{6r}\frac{1}{m_1^2}\left(\mathbf{L}^2 + 2\hbar^2\right) + \frac{\sigma}{2r}\frac{\mathbf{L}.\mathbf{s}_1}{m_1^2} - \frac{2\sigma}{3r}\frac{\mathbf{L}.\mathbf{s}_1}{m_1^2}. \end{aligned}$$

Notice the presence of new terms in $\mathbf{L}^2$ and $\mathbf{L}.\mathbf{s}_1$ with respect to the usual timelike vector potentials, due to the contributions of the moments of inertia of the straight-segment.

Two-particle nonrelativistic limit

$$
\begin{aligned}
H\Big|_{\mathrm{cm}} \quad = \quad & \frac{\mathbf{p}^2}{2\mu} + \sigma r - \frac{2\hbar\sigma}{\pi}\Big(\frac{1}{m_1} + \frac{1}{m_2}\Big) - \frac{1}{8}\Big(\frac{1}{m_1^3} + \frac{1}{m_2^3}\Big)(\mathbf{p}^2)^2 \\
& + \frac{\hbar^2}{4}\Big(\frac{1}{m_1^2} + \frac{1}{m_2^2}\Big)\frac{\sigma}{r} - \frac{\sigma}{6r}\Big(\frac{1}{m_1^2} + \frac{1}{m_2^2} - \frac{1}{m_1 m_2}\Big)(\mathbf{L}^2 + 2\hbar^2) \\
& + \frac{\sigma}{2r}\Big(\frac{\mathbf{L}.\mathbf{s}_1}{m_1^2} + \frac{\mathbf{L}.\mathbf{s}_2}{m_2^2}\Big) \\
& - \frac{2\sigma}{3r}\Big(\frac{1}{m_1^2} - \frac{1}{2m_1 m_2}\Big)\mathbf{L}.\mathbf{s}_1 - \frac{2\sigma}{3r}\Big(\frac{1}{m_2^2} - \frac{1}{2m_1 m_2}\Big)\mathbf{L}.\mathbf{s}_2.
\end{aligned}
$$

Absence of large-distance spin-spin forces. Additional contributions in the orbital angular momentum and spin-orbit terms coming from the moments of inertia of the straight-segment.

Nonrelativistic limit with the Wilson loop studied previously by several authors.

Eichten, Feinberg (1981); Gromes (1984);

Prosperi, Brambilla et al. (1988-1990);

Brambilla, Pineda, Soto, Vairo (2001).

Chiral symmetry breaking can be studied by incorporating into the previous equation the quark self-energy parts contained in the higher-order potentials.

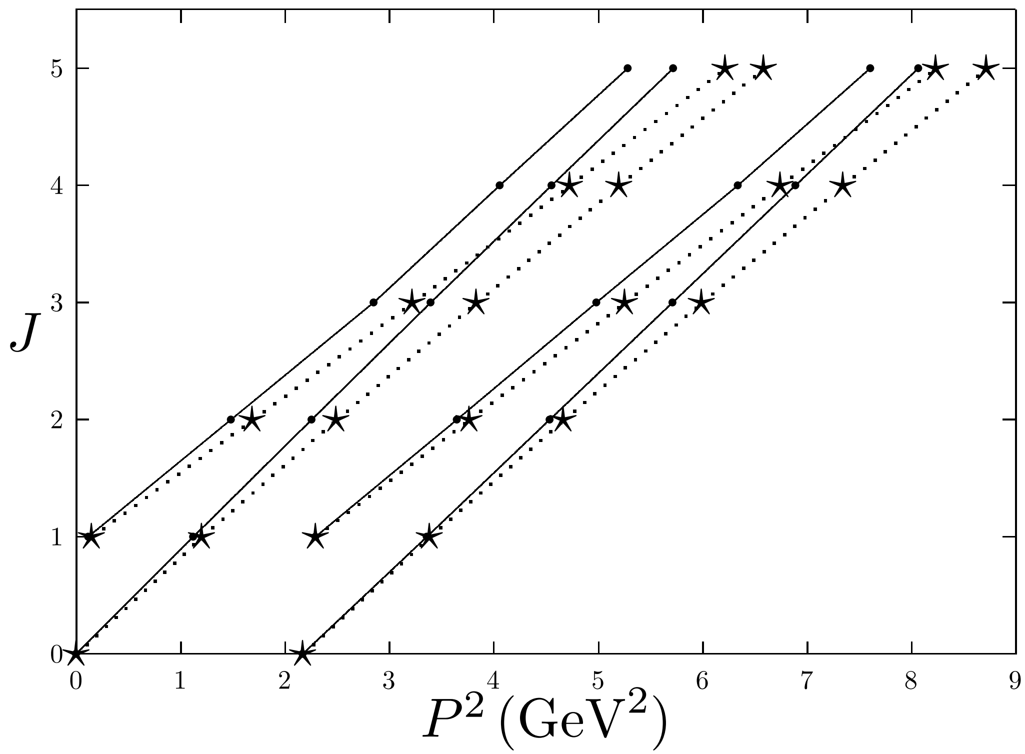
The resulting situation is very similar to that obtained in a theory with the exchange of a Coulomb-gluon. Self-energy equations similar to those studied previously by several authors:

Mandula et al. (1980-1984); Le Yaouanc et al. (1983-1985); Adler, Davis (1984); Alkofer, Amundsen (1988); Lagaë (1992).

Numerical values are, however, very small. One finds $< \overline{u}u > \simeq -(115\mathrm{MeV})^3$ and $F_\pi \simeq 15\mathrm{MeV}$, to be compared to the QCD sum rule prediction (Narison) $< \overline{u}u > \simeq -(225\mathrm{MeV})^3$ and the experimental value $F_\pi \simeq 94\mathrm{MeV}$.

Short-distance effects are not yet considered.

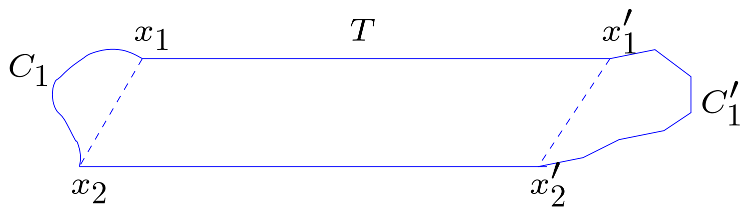
For light quarks, linear Regge trajectories are produced with slopes that are 15-20% larger than for the case of the pure timelike vector linear potential. The classical relationship between the Regge slope and the string tension, $\alpha' = (2\pi\sigma)^{-1}$, is enforced. (Olsson, 1989).



Does the type of the phase factor line C has any influence on the energy spectrum?

$\overline{\psi}(x_2)U(C_1;x_2,x_1)\psi(x_1)$

Problem easily studied in the static case. Quark propagators can be converted into phase factors along straight-lines along the time direction. One ends up with a Wilson loop with the following contour.



For finite time difference T, form of the minimal surface rather complicated. But in the limit $T \to \infty$, which displays the energy spectrum, the minimal surface shrinks to the union of the three independent minimal surfaces of the rectangle and of the two lateral contours. But the latter are independent of T.

$\Longrightarrow$ The energy spectrum is determined by the rectangle (dependent on T), while the lateral contours contribute only to the wave functions. Therefore, without loss of generality, one can consider straight lines for the phase factors joining the quarks to each other.

To study the spectra of hybrids (genuine gluoinic excitations), it is necessary to consider explicit gluon field strength insertions inside the phase factor lines:

$\overline{\psi}(x_2)U(C_2;x_2,z)F_{\mu\nu}(z)U(C_1;z,x_1)\psi(x_1).$