Conference Hadron Structure and QCD

Gatchina, 27 June - 1 July 2016

Gauge invariant approach to quark dynamics with polygonal Wilson lines

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Motivations

Gauge invariant objects expected to provide a more precise description of physical observables.
Better infrared behavior, free of spurious singularities.

Necessity of introducing gauge field path-ordered phase factors (parallel transporters) to ensure gauge invariance.
Gauge invariant quantities are generally extended objects $\Longrightarrow$ more complicated mathematical properties.

(Mandelstam, Nambu, Polyakov, Makeenko and Migdal.)

Method of approach

Functional method based on functional differentiation of path-ordered phase factors. One establishes functional relations between various Green's functions with different phase factor lines.

The kernels of the equations that are obtained are represented by Wilson loops, which are gauge invariant quantities. They are suitable to describe large-distance properties of QCD, since they are saturated at large distances by minimal surfaces, which satisfy the area law. (Makeenko and Migdal, 1980).

References: arXiv, 0709.0161, P.R. D 77 (2008) 045028; arXiv, 1003.5099, P.R. D 81 (2010) 114008; arXiv, 1304.0961, P.R. D 88 (2013) 025034; Phys. Part. Nucl. 45 (2014) 782.

Phase factor paths along polygonal lines

We work in the framework of QCD theory, with the color gauge group $SU(N_c)$, with the quark fields in the defining fundamental representation.

Paths along polygonal lines are of particular interest, since they can be decomposed into a succession of straight line segments. The latter have a Lorentz invariant form and have an unambiguous geometrical limit when their end points approach each other.

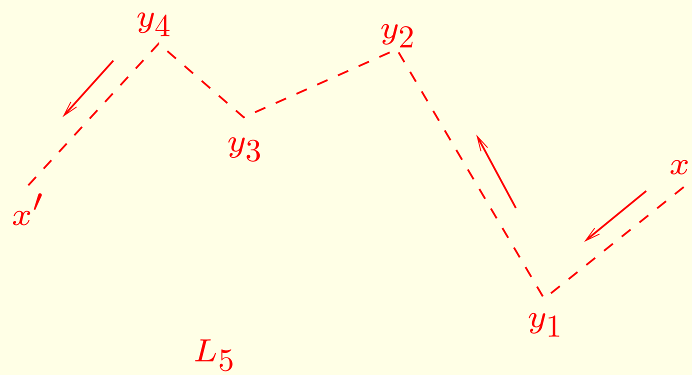
For our study, the polygonal lines form a complete set of paths for the quark Green's functions.

A phase factor with a single straight line segment going from x to y:

$$U(y,x) = Pe^{-ig\int_x^y dz^\mu A_\mu(z)}.$$

For a polygonal line L_n between the points x and x' with n segments and $(n-1)$ junction points y_1, y_2, $\ldots$, y_{n-1}, one has

$$U(x',x;L_n) = U(x',y_{n-1})U(y_{n-1},y_{n-2})\ldots U(y_2,y_1)U(y_1,x).$$



Rigid path derivation

For a rigid straight line segment a displacement of one end of the segment generates also displacements of the interior points of the segment with appropriate weights. (Mandelstam, 1968.)

$$\frac{\partial U(y,x)}{\partial y^{\alpha}} = -igA_{\alpha}(y)U(y,x) + ig(y-x)^{\beta}\int_0^1 d\lambda\,\lambda\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\partial U(y,x)}{\partial x^{\alpha}} = +igU(y,x)A_{\alpha}(x) + ig(y-x)^{\beta}\int_0^1 d\lambda\,(1-\lambda)\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Conventions to represent the contributions of the integrals:

$$\frac{\bar{\delta}U(y,x)}{\bar{\delta}y^{\alpha+}} \equiv ig(y-x)^{\beta}\int_0^1 d\lambda\,\lambda\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0),$$

$$\frac{\bar{\delta}U(y,x)}{\bar{\delta}x^{\alpha-}} \equiv ig(y-x)^{\beta}\int_0^1 d\lambda\,(1-\lambda)\,U(1,\lambda)F_{\beta\alpha}(\lambda)U(\lambda,0).$$

Gauge invariant Green's functions along polygonal lines

Gauge invariant Green's functions with polygonal lines can be classified according to the number of segments they contain.

The gauge invariant two-point quark Green's function with a polygonal line with n segments and $n-1$ junction points y_1, y_2, . . .,y_{n-1} between the segments is defined as

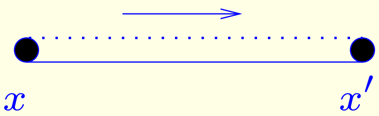
$$S_{(n)}(x, x'; y_{n-1}, \dots, y_1) = -\frac{1}{N_c}\langle\overline{\psi}(x')U(x', y_{n-1})\dots U(y_1, x)\psi(x)\rangle,$$

where each U is along a straight line segment.

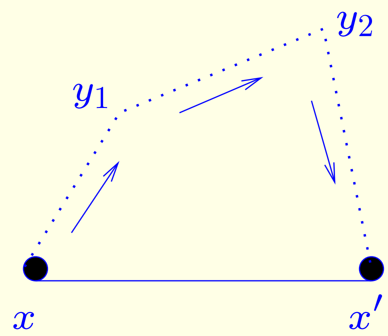
For one straight line, one has:

$$S_{(1)}(x, x') \equiv S(x, x') = -\frac{1}{N_c}\langle\overline{\psi}(x')\,U(x', x)\,\psi(x)\rangle.$$

Pictorially:



$$S(x,x') \equiv S_{(1)}(x,x') = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',x)\, \psi(x) >$$



$$S_{(3)}(x,x';y_2,y_1) = -\frac{1}{N_c} < \overline{\psi}(x')\, U(x',y_2)U(y_2,y_1)U(y_1,x)\, \psi(x) >$$

Wilson loops

$$\Phi(C) = \frac{1}{N_c} \mathrm{tr} P e^{-ig \oint_C dx^\mu A_\mu(x)}.$$

Vacuum expectation value:

$$W(C) = \langle \Phi(C) \rangle.$$

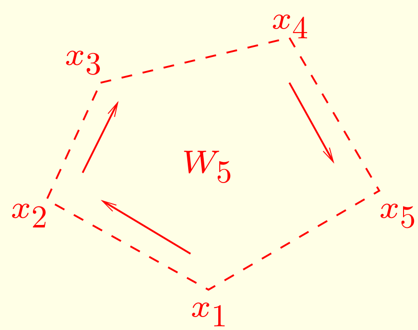
Functional representation:

$$W(C) = e^{F(C)}.$$

(Dotsenko and Vergeles, Makeenko and Migdal, 1980.)

If the contour C is a polygon C_n with n sides and n successive junction points $x_1,\ x_2,\ \ldots,\ x_n$, then we write:

$$W(x_n, x_{n-1}, \ldots, x_1) = W_n = e^{F_n(x_n, x_{n-1}, \ldots, x_1)} = e^{F_n}.$$



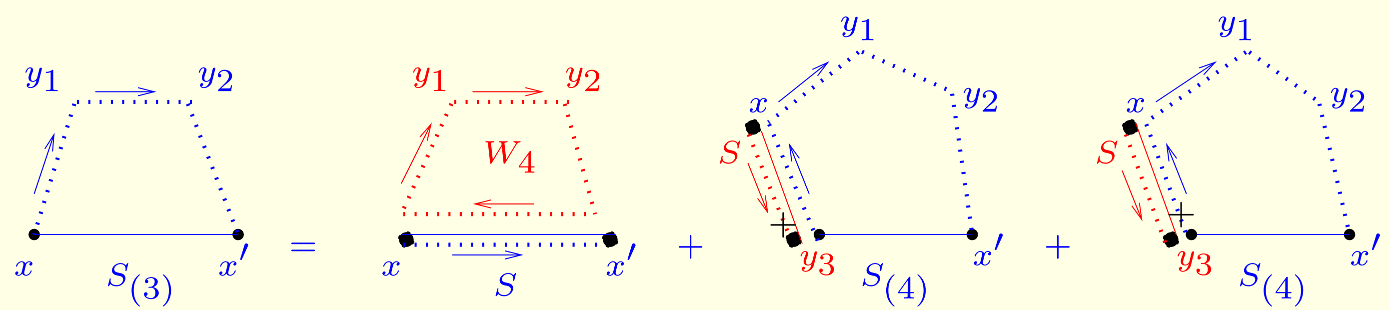
$W_5(x_5, x_4, \ldots, x_1) = e^{F_5(x_5, \ldots, x_1)}$

Functional relations for Green's functions

Use of equations of motion of quark fields and integrations yield functional relations between the various Green's functions with polygonal lines. (Integration symbols omitted.)

$$S_{(n)}(x,x';y_{n-1},\ldots,y_1) = S(x,x')\,e^{F_{n+1}(x',y_{n-1},\ldots,y_1,x)}$$

$$+\left(\frac{\bar{\delta}S(x,y_n)}{\bar{\delta}y_n^{\alpha+}} + S(x,y_n)\frac{\bar{\delta}}{\bar{\delta}y_n^{\alpha-}}\right)\gamma^{\alpha}\,S_{(n+1)}(y_n,x';y_{n-1},\ldots,y_1,x).$$

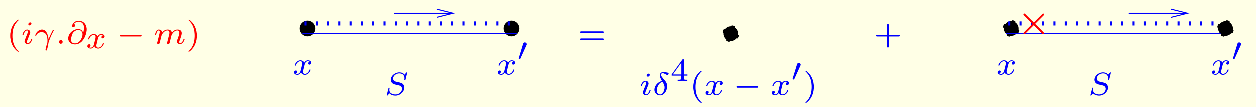


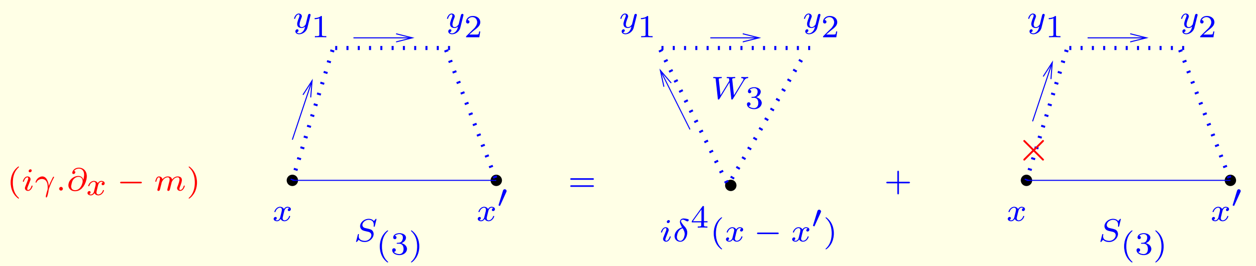
$\Longrightarrow$ S is the only dynamically independent gauge invariant quark Green's function.

Equations of motion of Green's functions

$$(i\gamma.\partial_{(x)}-m)S_{(n)}(x,x';y_{n-1},\ldots,y_1) = i\delta^4(x-x')e^{F_n(x,y_{n-1},\ldots,y_1)}$$

$$+i\gamma^{\mu}\frac{\bar{\delta}S_{(n)}(x,x';y_{n-1},\ldots,y_1)}{\bar{\delta}x^{\mu-}}.$$





Integrodifferential equation for the two-point function

S satisfies the following equation of motion:

$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu \frac{\bar{\delta}S(x,x')}{\bar{\delta}x^{\mu-}}.$$

The rigid path derivative $\bar{\delta}S(x,x')/\bar{\delta}x^{\mu-}$ is calculated using the functional relations between Green's functions. One establishes the following integrodifferential equation for the Green's function $S(x,x')$:

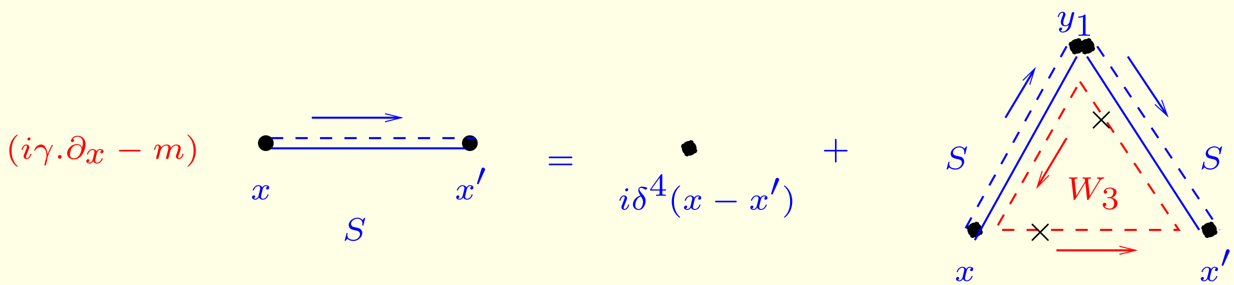
$$(i\gamma.\partial_{(x)} - m)S(x,x') = i\delta^4(x-x') + i\gamma^\mu \Big\{ K_{2\mu}(x',x,y_1)\, S_{(2)}(y_1,x';x)$$
$$+ \sum_{n=3}^{\infty} K_{n\mu}(x',x,y_1,\ldots,y_{n-1})\, S_{(n)}(y_{n-1},x';x,y_1,\ldots,y_{n-2}) \Big\}.$$

The kernel K_n contains globally n derivatives of Wilson loops with a $(n+1)$-sided polygonal contour and also the Green's function S and its derivative.

The dominant parts of the kernel are expected to be those containing the least number of derivatives of Wilson loops.

Thus the leading term is the second-order term (the first-order one being zero for symmetry reasons).

$$\frac{\bar{\delta} S(x,x')}{\bar{\delta} x^{\mu -}} \simeq -\int d^4y_1 \, \frac{\bar{\delta}^2 F_3(x',x,y_1)}{\bar{\delta} x^{\mu -} \bar{\delta} y_1^{\alpha_1 +}} \, e^{F_3(x',x,y_1)} \, S(x,y_1) \, \gamma^{\alpha_1} \, S(y_1,x').$$



Two-dimensional QCD

Confinement properties of two-dimensional QCD at large N_c are very similar to those of QCD in four dimensions; provides a simplified framework for qualitative and quantitative investigations.

Crossed diagrams and quark loop contributions disappear. ('t Hooft, 1974.)
Wilson loop averages are exponential functionals of the areas of the surfaces enclosed by the contours. (Kazakov and Kostov, Bralić, 1980.) The area law naturally produced.
The second-order derivative of the logarithm of the Wilson loop average is a delta-function. Kernels with more derivatives than two disappear.
$\Longrightarrow$ Equation of S with the lowest-order kernel becomes an exact equation.

$$(i\gamma.\partial - m)S(x) = i\delta^2(x) - \sigma\gamma^\mu(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})x^\nu x^\beta$$

$$\times\Big[\int_0^1 d\lambda\lambda^2 S((1-\lambda)x)\gamma^\alpha S(\lambda x) + \int_1^\infty d\xi S((1-\xi)x)\gamma^\alpha S(\xi x)\Big].$$

(σ is the string tension.)

The interaction term is quasi-local in x: x is integrated along a line and not in two dimensions. $\Longrightarrow$ the equation can also be studied in instantaneous type limits.

Resolution of the equation

Passing to momentum space:

$$S(p) = \gamma.pF_1(p^2) + F_0(p^2).$$

The equation of S can then be studied through the singularities of S that may be present. The problem can be solved explicitly in analytic form.

The covariant functions $F_1(p^2)$ and $F_0(p^2)$ have an infinite number of branch points at mass values $M_1^2,\ M_2^2, \ldots,\ M_n^2, \ldots$, ordered with increasing values; in particular $M_1 > m$. The power of the singularity is $-3/2$.

The solutions are:

$$F_1(p^2) = -i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n \frac{1}{(M_n^2 - p^2)^{3/2}},$$

$$F_0(p^2) = i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty}(-1)^n b_n \frac{M_n}{(M_n^2 - p^2)^{3/2}},$$

or for S,

$$S(p) = -i\frac{\pi}{2\sigma}\sum_{n=1}^{\infty} b_n \frac{(\gamma.p + (-1)^{n+1}M_n)}{(M_n^2 - p^2)^{3/2}}.$$

The M_ns and b_ns satisfy algebraic equations that are solved numerically.
For large n:

$$M_n^2 \simeq \sigma\pi n, \quad b_n \simeq \frac{\sigma^2}{M_n}, \quad \text{for } \sigma\pi n \gg m^2.$$

Asymptotic behaviors:

$$F_1(p^2) \underset{|p^2|\to\infty}{=} \frac{i}{p^2},$$

$$F_0(p^2) \underset{|p^2|\to\infty}{=} \frac{im}{p^2}, \qquad m \neq 0,$$

$$F_0(p^2) \underset{|p^2|\to\infty}{=} \frac{2i\sigma}{N_c} \frac{\langle \overline{\psi}\psi \rangle}{(p^2)^2}, \qquad m = 0.$$

(Politzer, 1976.)

Conclusion

1) The equations satisfied by gauge invariant quark Green's functions may provide complementary informations about the spectral properties of quark and gluon fields, not available with ordinary Green's functions.

2) In two-dimensional QCD at large-N_c, the spectral functions are infrared finite and lie on the positive real axis of p^2. No singularities in the complex plane or on the negative real axis have been found. $\Longrightarrow$ Quarks contribute with positive energies.

3) The singularities are represented by an infinite number of threshold type singularities, characterized by positive masses M_n $(n = 1, 2, \ldots)$. The corresponding singularities are stronger than simple poles and this feature might prevent observability of quarks as free particles.

4) The threshold masses M_n represent dynamically generated masses and maintain the scalar part of the Green's function at a nonzero value.