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Energy and decay width of the πK atom

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Hadronic atoms:

Bound states of hadrons, formed by Coulomb attraction.

$$(\pi^- p),\ (\pi^- d),\ (\pi^+\pi^-),\ (\pi^+ K^-),\ (K^- p),\ (K^- d).$$

Because of electromagnetic mass differences, these atoms decay by strong interaction into isospin partners.

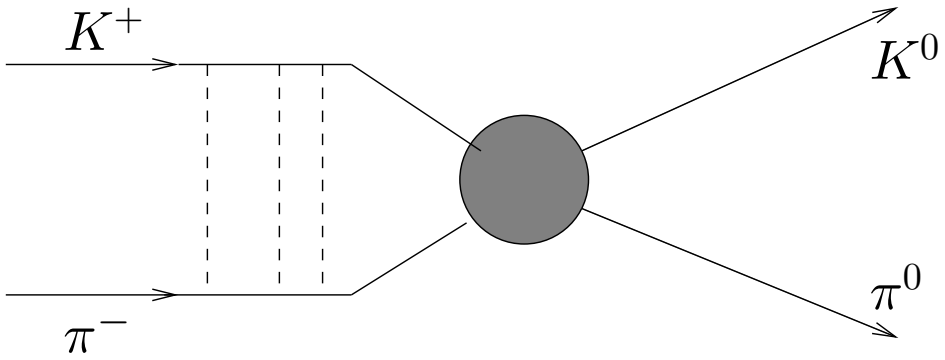
$$(\pi^- p) \to \pi^0 n,$$

$$(\pi^+\pi^-) \to \pi^0\pi^0,$$

$$(\pi^+ K^-) \to \pi^0 \overline{K}^0,$$

$$(K^- p) \to \overline{K}^0 n,\ \ \Lambda\pi^0,\ \ \Sigma^0\pi^0.$$

For the πK atom, one has the process:



Because of the instability of the π and K mesons, it is not possible to do direct low-energy $\pi K \to \pi K$ scattering experiments.

Generally, on experimental grounds, the scattering lengths are reconstituted by extrapolating high-energy data (above 0.9 GeV) down to threshold. Big uncertainties.

Here, the decay widths and energy shifts of the πK atom states give us direct information on the $\pi K \to \pi K$ scattering lengths.

Theoretical interest of the scattering lengths

They are related to order parameters of spontaneous breaking of chiral symmetry in QCD. They give in particular information about the quark condensate and through it on the mechanism of chiral symmetry breaking and about the role of quark flavors in it.

The scattering lengths are calculable in Chiral Perturbation Theory.

In the absence of electromagnetism and in the isospin symmetry limit, two isospin-fixed, $I = 1/2,\ 3/2$, scattering lengths with each orbital angular momentum ℓ. We are mainly concerned with the S-wave scattering lengths $a_{\ell=0}^{I=1/2}$ and $a_{\ell=0}^{I=3/2}$. One also defines the isospin even and odd combinations:

$$a_0^+ = \frac{1}{3}(a_0^{1/2} + 2a_0^{3/2}), \qquad a_0^- = \frac{1}{3}(a_0^{1/2} - a_0^{3/2}).$$

Summary of ChPT (chiral perturbation theory) predictions.

Authors	ChPT	$m_\pi a_0^{1/2}$	$m_\pi a_0^{3/2}$
Weinberg (1966)	Tree	0.14	-0.07
Bernard *et al.* (1991)	One-loop	0.19 ± 0.02	-0.05 ± 0.02
Bijnens *et al.* (2004)	Two-loop	0.220	-0.047

An extrapolation of high-energy experimental data, using Roy and Steiner equations (dispersion relations, crossing symmetry and partial wave decomposition), has been done by Büttiker, Descotes-Genon and Moussallam (2004). The results are:

$$m_\pi a_0^{1/2} = 0.224 \pm 0.022, \qquad m_\pi a_0^{3/2} = -(0.448 \pm 0.077) \times 10^{-1}.$$

Convergence of the theoretical calculations towards the experimental values.

Deser *et al.* formulas

Using the short-range character of the strong interaction, Deser *et al.* (1954) have obtained formulas in the nonrelativistic limit for the decay width and energy shift of hadronic atoms (with quantum numbers (n, ℓ)).

$$
\begin{aligned}
E_{n,\ell}^{(0)} &= -\frac{\mu\alpha^2}{2n^2} \qquad \text{(Coulomb binding energy)}, \\
\Delta E_{n,\ell} &= -2\mu^2 \frac{\alpha^3}{n^3} \left(a_0^+ + a_0^-\right) \delta_{\ell,0} \qquad \text{(energy shift)}, \\
\Gamma_{n,\ell} &= 8\, p_{n0}^*\, \mu^2 \frac{\alpha^3}{n^3} \left(a_0^-\right)^2 \delta_{\ell,0} \qquad \text{(decay width)}.
\end{aligned}
$$

μ: reduced mass of the πK system.

p_{n0}^*: the c.m. momentum of the neutral mesons $(\pi^0 K^0)$ after the decay of the bound state with quantum numbers $(n, 0)$.

Experimental measurement of the lifetime and of the energy shift allow us to obtain the values of the scattering lengths.

Numerically:

$$E_{10}^0 = -2898.61\ \mathrm{eV}, \qquad \tau_{10} = (3.7 \pm 0.5) \times 10^{-15}\ \mathrm{s},$$

$$\Delta E_{10} = -8.86\ \mathrm{eV}, \qquad \Delta E_{20} = -1.11\ \mathrm{eV},$$

$$E_{21} - E_{20} = 1.11\ \mathrm{eV}.$$

However, these predictions do not include relativistic corrections, neither electromagnetic radiative corrections, nor isospin symmetry breaking effects. Corrections of a few per cent expected. In pionium $(\pi^+\pi^-)$, there was found 6% corrections in the decay width.

Necessity of evaluating higher-order corrections to the Deser *et al.* formulas to have a precise test of ChPT.

A similar problem met with the pionium $(\pi^+\pi^-)$. Many theoretical works on the subject. DIRAC experiment at CERN.

For the (πK) atom, experimental projects at CERN, GSI and J-PARC.

Theoretical calculations done by J. Schweizer (2004), using the nonrelativistic effective theory approach developed by Caswell and Lepage (1986).

We analyse the problem with the constraint theory - quasipotential method approach.

- Bound state formalism,
- πK system.

Bound state formalism

Constraint theory (Dirac, 1964) allows, through the use of first-class constraints, the elimination of redundant variables, respecting at the same time the symmetries of the theory (in the present case the Poincaré invariance). Relative energies and relative times of particles of multiparticle systems should not play a dynamical role in relativistic theories. For a two-particle system, with momenta p_1 and p_2 and physical masses m_1 and m_2, the following constraint eliminates the relative energy in a covariant way:

$$C(P,p) \equiv (p_1^2 - p_2^2) - (m_1^2 - m_2^2) = 0, \qquad P = p_1 + p_2, \quad p = \frac{1}{2}(p_1 - p_2).$$

Also respects the symmetry between the two particles and remains valid on the mass shell and in the free case.

Two spin-0 particles, 1 and 2. T: the scattering amplitude of the process $1+2 \to 1'+2'$. We define:

$$\widetilde{T} = \frac{i}{2\sqrt{s}} T \Big|_C .$$

(Constraint C applied on external momenta of T. $s = P^2$.)

Postulate: $\widetilde{T}$ satisfies, by means of an effective propagator g_0, a three-dimensional Lippmann–Schwinger type equation leading to the definition of a kernel or a potential V:

$$V = \widetilde{T} - V\, g_0\, \widetilde{T}.$$

The expression of g_0 is chosen so that V is hermitian in the elastic unitarity region.

Constraint C implies equality of the Klein-Gordon operators of particles 1 and 2:

$$H_0(s,\mathbf{p}) \equiv (p_1^2 - m_1^2)\big|_C = (p_2^2 - m_2^2)\big|_C = b_0^2(s) - \mathbf{p}^2,$$
$$b_0^2(s) \equiv \frac{s}{4} - \frac{1}{2}(m_1^2 + m_2^2) + \frac{(m_1^2 - m_2^2)^2}{4s}.$$

(Written in the c.m. frame.) g_0 is chosen as the propagator associated with these operators:

$$g_0(s,\mathbf{p}) = \frac{1}{H_0(s,\mathbf{p}) + i\varepsilon} = \frac{1}{b_0^2(s) - \mathbf{p}^2 + i\varepsilon}.$$

The Lippmann-Schwinger type equation $V = \widetilde{T} - V\, g_0\, \widetilde{T}$ defines an iterative series for V, where the integrations are three-dimensional. In addition to the usual Feynman diagrams of T, one has three-dimensional diagrams in which the constraint C is used ("Constraint diagrams"). If

$$\widetilde{T} = \widetilde{T}_1 + \widetilde{T}_2 + \widetilde{T}_3 + \cdots ,$$

then

$$V = \widetilde{T}_1 + \widetilde{T}_2 - \widetilde{T}_1 g_0 \widetilde{T}_1 + \cdots \ .$$

If T has a pole in s, one deduces from the above equation a three-dimensional covariant bound state equation:

$$\Big[\, g_0^{-1} - V \,\Big]\Psi = 0.$$

Perturbation theory

$$V = V_1 + V_2,$$

such that the solutions associated with V_1 are known: eigenvalues $s_n^{(0)}$ and wave functions φ_n. G_1 is the Green function associated with V_1:

$$G_1 = g_0 + g_0\, V_1\, G_1 = G_1' + \frac{\varphi_n \varphi_n^\dagger}{(s - s_n^{(0)} + i\varepsilon)}.$$

The complete Green function G satisfies the equation

$$G = G_1 + G_1 V_2 G.$$

One obtains a perturbative series for the complete eigenvalue s_n in terms of matrix elements of V_2:

$$s_n = s_n^{(0)} + \left\{ (\varphi_n^\dagger \, V_2 \, \varphi_n) + (\varphi_n^\dagger \, V_2 \, G_1' \, V_2 \, \varphi_n) + (\varphi_n^\dagger \, V_2 \, \varphi_n)(\varphi_n^\dagger \, \frac{\partial V_2}{\partial s} \, \varphi_n) \right\} \Big|_{s=s_n^{(0)}}$$

$$+ \cdots .$$

Lepage (1977), Bodwin and Yennie (1978), Gasser *et al.* (2001).

πK system

Treat the πK system by means of a coupled channel formalism including the charged sector (c) made of $\pi^- K^+$ and the neutral sector (n) made of $\pi^0 K^0$.

Because of the decay process $\pi^- K^+ \rightarrow \pi^0 K^0$, the energy of the bound state becomes complex with a negative imaginary part. The scattering amplitudes and Green functions involving the above sectors have a common pole at the position of the complex energy of the bound state. We introduce a two-component wave function Ψ as:

$$\Psi = \begin{pmatrix} \Psi_c \\ \Psi_n \end{pmatrix}.$$

The potential V is defined in matrix form in the corresponding space:

$$V = \begin{pmatrix} V_{cc} & V_{cn} \\ V_{nc} & V_{nn} \end{pmatrix}.$$

The iteration effective propagator g_0 is now composed of two propagators:

$$g_0 = \begin{pmatrix} g_{0c} & 0 \\ 0 & g_{0n} \end{pmatrix}.$$

g_{0c} and g_{0n} are defined with the physical masses of the charged and neutral particles, respectively.

The wave equation takes now the form of two coupled equations:

$$\begin{aligned}\left(g_{0c}^{-1} - V_{cc}\right)\Psi_c - V_{cn}\Psi_n &= 0,\\ -V_{nc}\Psi_c + \left(g_{0n}^{-1} - V_{nn}\right)\Psi_n &= 0.\end{aligned}$$

The wave function Ψ_n represents an outgoing wave created by the charged state; it can be eliminated in favor of Ψ_c, yielding the wave equation for the latter wave function:

$$g_{0c}^{-1}\Psi_c = V_{cc}\Psi_c + V_{cn}\left(1 - g_{0n}V_{nn}\right)^{-1} g_{0n}V_{nc}\Psi_c.$$

This is the bound state equation describing the properties of the πK atom.

The potentials V are calculated from the Lippmann–Schwinger type equation, written now in matrix form in terms of the scattering amplitudes of the processes $\pi^- K^+ \to \pi^- K^+$, $\pi^- K^+ \to \pi^0 K^0$, $\pi^0 K^0 \to \pi^- K^+$, $\pi^0 K^0 \to \pi^0 K^0$.

When electromagnetism and isospin symmetry breaking are switched-off, one remains with the strong interaction or hadronic amplitudes $\widetilde{T}_h$ in the isospin symmetry limit; these are related to the isospin invariant amplitudes $\widetilde{T}^I$ from which one defines the strong interaction scattering lengths $a_\ell^{1/2}$ and $a_\ell^{3/2}$ that we have met.

Perturbation theory

Perturbation theory is developed by considering the Coulomb potential as the zeroth-order potential:

$$V_{cc} = 2\mu V_C + \overline{V}_{cc}, \qquad V_C = -\frac{\alpha}{r}.$$

The (complex) expression of the energy shift is:

$$2\mu\Delta\mathcal{E}_n = \Big\{ \varphi^\dagger_{n\ell}\,\overline{V}_{cc}\,\varphi_{n\ell} + \varphi^\dagger_{n\ell}\,V_{cn}\,g_{0n}\,V_{nc}\,\varphi_{n\ell} + \varphi^\dagger_{n\ell}\,\overline{V}_{cc}\,\frac{G'_C}{2\mu}\,\overline{V}_{cc}\,\varphi_{n\ell}$$
$$+\varphi^\dagger_{n\ell}\Big(\overline{V}_{cc}\,\frac{G'_C}{2\mu}\,V_{cn}\,g_{0n}\,V_{nc} + V_{cn}\,g_{0n}\,V_{nc}\,\frac{G'_C}{2\mu}\,\overline{V}_{cc}\Big)\varphi_{n\ell}\Big\}\Big|_{\mathcal{E}=E_n^{(0)}}$$
$$-\Big(\frac{m^2_{\pi^-} + m^2_{K^+} - m_{\pi^-}m_{K^+}}{(m_{\pi^-} + m_{K^+})^2}\Big)E_n^{(0)2}.$$

Orders of magnitude of various terms are then evaluated by using the counting rules of QED bound states.

Coulomb bound state energies are of order α^2.

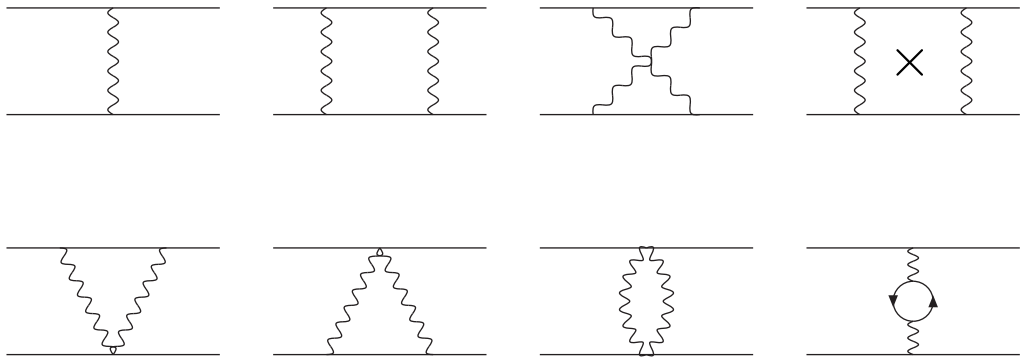
Next corrections come from strong interaction, which gives terms of order α^3; these correspond to the Deser *et al.* formulas.

We are interested by the $O(\alpha)$ corrections to the latter ($O(\alpha^4)$ terms). They originate from three sources:

- Pure electromagnetic corrections.
- Interference terms between strong interaction and electromagnetism, including isospin symmetry violation.
- Second-order effects of perturbation theory.

Electromagnetic interaction

Pure electromagnetic corrections arise in the channel $\pi^- K^+ \to \pi^- K^+$ from one- and two-photon exchange diagrams and also include vacuum polarization contribution.



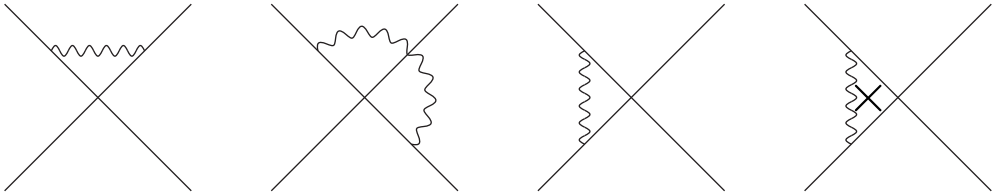
Contribute only to the real part of the energy shift. Most important contribution from vacuum polarization (25% of the strong interaction effect of order α^3).

Strong interaction in the presence of electromagnetism

Strong interaction effects calculated with the aid of the chiral effective Lagrangian. The latter is built as a low-energy perturbation expansion in the external momenta of the Goldstone bosons (π, K, η) and the quark masses. At each order of the perturbation series, renormalization introduces new coupling constants, called low-energy constants, which are in general measurable quantities.

Strong interaction in the presence of electromagnetism and isospin symmetry breaking was considered by Urech (1995). One dozen of new low-energy constants, which are still poorly known and are the source of uncertainties. One considers them through their order of magnitude ($1/(16\pi^2)$).

Typical diagrams with one photon exchange are:



Were evaluated by Nehme and Talavera (2002) and Kubis and Meissner (2002). The shifts are:

$$\Delta E_{h\gamma,n0} = (1.1 \pm 3.2) \times 10^{-2} \times E^{(1)}_{h,n0},$$
$$\Delta \Gamma_{h\gamma,n0} = (2.4 \pm 1.4) \times 10^{-2} \times \Gamma^{(1)}_{h,n0}.$$

It is hoped that more accurate evaluations of the electromagnetic low-energy constants (with sum rules and resonance saturations) will allow one in the future to reduce substantially the above uncertainties.

Second order of perturbation theory

Main contributions come from hadronic-hadronic type correlators and from hadronic-vacuum polarization type interferences.

In total of the order of 1% in the real energy shift and 2% in the decay width.

Our results are in agreement with those found by J. Schweizer.

Summary of results

We collect the three types of correction: pure electromagnetism, strong interaction in the presence of electromagnetism, second-order of perturbation theory.

The decay width of the ground state is:

$$\Gamma_{10} = 8 p_{10}^{*} \mu^2 \alpha^3 \left(a_0^-\right)^2 \left(1 + 0.046 \pm 0.014\right).$$

The energy splitting between the $2P$ and $2S$ states is:

$$\left(E_{21} - E_{20}\right) = \tfrac{1}{4} \mu^2 \alpha^3 \left(a_0^+ + a_0^-\right) \left(1 + 0.023 \pm 0.032\right) + 0.29\,(\mathrm{eV}).$$

These formulas allow one to extract from the experimental results on the decay width and the energy splitting the values of the strong interaction scattering lengths a_0^- and a_0^+.